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QUANTITATIVE SUB-BALLISTICITY OF SELF-AVOIDING WALK ON THE HEXAGONAL LATTICE

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We prove quantitative sub-ballisticity for the self-avoiding walk on the hexagonal lattice. Namely, we show that, with high probability, a self-avoiding walk of length n does not exit a ball of radius $O(n/\log n)$. Previously, only a nonquantitative $o(n)$ bound was known from the work of Duminil-Copin and Hammond (*Comm. Math. Phys.* **324** (2013) 401–423). As an important ingredient of the proof, we show that at criticality the partition function of bridges of height T decays polynomially fast to 0 as T tends to infinity, which we believe to be of independent interest.

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THE STOCHASTIC JACOBI FLOW

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The problem of conditioning on the occupation field was investigated for the Brownian motion in 1998 independently by Aldous (*Electron. Commun. Probab.* **3** (1998) 79–90) and Warren and Yor (In *Séminaire de Probabilités, XXXII* (1998) 328–342, Springer) and recently for the loop soup at intensity $1/2$ by Werner (In *Séminaire de Probabilités XLVIII* (2016) 481–503 Springer), Sabot and Tarrès (*Probab. Theory Related Fields* **165** (2016) 559–580), and Lupu, Sabot and Tarrès (*Electron. J. Probab.* **24** (2019) 1–28). We consider this problem in the case of the Brownian loop soup on the real line, and show that it is connected with a flow version of Jacobi processes, called Jacobi flow. We show that such flows naturally appear in the setting of the perturbed reflecting Brownian motion. The Jacobi flow is also related to Fleming–Viot processes, as established by Bertoin and Le Gall (*Probab. Theory Related Fields* **126** (2003) 261–288, *Ann. Inst. Henri Poincaré Probab. Stat.* **41** (2005) 307–333) and Dawson and Li (*Ann. Probab.* **40** (2012) 813–857). This relation allows us to interpret Perkins’ disintegration theorem between Feller continuous state branching-processes and Fleming–Viot processes as a decomposition of Gaussian measures. Our approach gives a unified framework for the problems of disintegrating on the real line. The connection with Bass–Burdzy flows which was drawn in Warren (*Probab. Theory Related Fields* **133** (2005) 559–572) and Lupu, Sabot and Tarrès (*Electron. J. Probab.* **26** (2021) 1–25) is shown to be valid in the general case.

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LARGE DEVIATIONS OF THE SCHWARZIAN FIELD THEORY

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We prove a large deviations principle for the probabilistic Schwarzian Field Theory at low temperatures. We demonstrate that the good rate function is equal to the action of the Schwarzian Field Theory, and we find its minimisers. In addition, we define an analogue of the Hölder condition on the functional space $\text{Diff}^1(\mathbb{T})/\text{PSL}(2, \mathbb{R})$ in terms of cross-ratio observables, characterise them in terms of the usual Hölder property on the space of continuous functions, and deduce the corresponding compact embedding theorem. We also show that the Schwarzian measure concentrates on functions satisfying the defined condition.

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NONRIGIDITY PROPERTIES OF THE COULOMB GAS

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We prove existence of infinite volume d -dimensional Coulomb gases which are not number rigid for $d \geq 3$. This makes the Coulomb gas the Gibbs point process with the longest range pairwise interaction (i.e., with the smallest s in the interaction kernel $g(x) = |x|^{-s}$) for which number nonrigidity has been proved in $d \geq 3$. We rule out properties stronger than number rigidity for the two-dimensional Coulomb gas.

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THE KPZ EQUATION AND THE DIRECTED LANDSCAPE

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This paper proves the convergence of the narrow wedge solutions of the KPZ equation to the Airy sheet and the directed landscape. This is the first convergence result to the Airy sheet and the directed landscape established for a positive temperature model. We also give an independent proof for the convergence of the KPZ equation to the KPZ fixed point for general initial conditions in the locally uniform topology. Together with the directed landscape convergence, we show the joint convergence to the KPZ fixed point for multiple initial conditions.

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SPATIAL PARTICLE PROCESSES WITH COAGULATION: GIBBS-MEASURE APPROACH, GELATION, AND SMOLUCHOWSKI EQUATION

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We study a spatial Markovian *particle system with pairwise coagulation*, a spatial version of the Marcus–Lushnikov process: according to a *coagulation kernel* K , particle pairs merge into a single particle, and their masses are united. We introduce a *statistical-mechanics approach* to the study of this process. We derive an explicit formula for the empirical process of the particle configuration at a given fixed time T in terms of a reference Poisson point process, whose points are trajectories that coagulate into one particle by time T . The noncoagulation between any two of them induces an exponential pair-interaction, which turns the description into a *many-body system with a Gibbsian pair-interaction*.

Based on this, we first give a *large-deviation principle* for the joint distribution of the particle histories (conditioning on an upper bound for particle sizes), in the limit as the number N of initial atoms diverges and the kernel scales as $\frac{1}{N}K$. We characterise the minimiser(s) of the rate function, we give criteria for its uniqueness and prove a *law of large numbers* (unconditioned). Furthermore, we use the unique minimiser to construct a solution of the *Smoluchowski equation* and give a criterion for the occurrence of a *gelation phase transition*.

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FLUCTUATIONS OF THE ATLAS MODEL FROM INHOMOGENEOUS STATIONARY PROFILES

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The infinite Atlas model describes the evolution of a countable collection of Brownian particles on the real line, where the lowest particle is given a drift of $\gamma \in [0, \infty)$. The case $\gamma = 0$ is referred to as the Harris model. In this work we study equilibrium fluctuations for the Atlas model for $\gamma \in [0, \infty)$ when the system of particles starts from an inhomogeneous stationary profile with exponentially growing density. We show that the appropriately centered and scaled occupation measure of the particle positions, with suitable translations, viewed as a space-time random field, converges to a limit that can be characterized in terms of a certain stochastic partial differential equation (SPDE). The initial condition for this equation is given by a Brownian motion, the equation is driven by an additive space-time noise that is white in time and colored in space, and the linear operator governing the evolution is the infinitesimal generator of a geometric Brownian motion. We use this SPDE to also characterize the fluctuations of the ranked particle positions with a suitable centering and scaling. Our results describe the behavior of the particles in the bulk, and one finds that the Gaussian process describing the asymptotic fluctuations has the same Hölder regularity as a fractional Brownian motion with Hurst parameter $1/4$. This connection with a fractional Brownian motion becomes even more exact when the inhomogeneous stationary profiles approach the homogeneous stationary profile in a suitable manner. One finds that, unlike the setting of a homogeneous profile (Dembo and Tsai (*Ann. Probab.* **45** (2017) 4529–4560)), the behavior on the lower edge of the particle system is very different from the bulk behavior, and in fact, the variance of the Gaussian limit diverges to ∞ as one approaches the lower edge. Indeed, our results show that, with the gaps between particles given by one of the inhomogeneous stationary distributions, the lowest particle, started from 0, with a linear in time translation, converges in distribution to an explicit non-Gaussian limit as $t \rightarrow \infty$. In the special case of the Harris model, this limit is given as the law of the difference of two independent Gumbel distributions.

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EFFECTIVE MASS OF THE FRÖHLICH POLARON AND THE LANDAU–PEKAR–SPOHN CONJECTURE

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We prove that there is a constant $\bar{C} \in (0, \infty)$ such that the effective mass $m(\alpha)$ of the Fröhlich Polaron satisfies $m(\alpha) \geq \bar{C}\alpha^4$, which is sharp according to a long-standing prediction of Landau–Pekar (*Zh. Eksp. Teor. Fiz.* **18** (1948) 419–423) from 1948 and of Spohn (*Ann. Physics* **175** (1987) 278–318) from 1987.

The method of proof, which demonstrates how the sharp quartic divergence rate of $m(\alpha)$ appears in a natural way, is based on analyzing the Gaussian representation of the Polaron measure and that of the associated tilted Poisson point process developed in (*Comm. Pure Appl. Math.* **73** (2020) 350–383). Additionally, our technique here leads to accompanying results including: (1) an explicit identification of local interval process from (*Comm. Pure Appl. Math.* **73** (2020) 350–383) in the strong coupling limit in terms of functionals of the Pekar process (*Ann. Probab.* **48** (2020) 2119–2144), (2) strict monotonicity of the effective mass $m(\alpha)$ for all $\alpha > 0$ and (3) the quartic divergence of $m(\alpha)$ for a generalized class of Polaron type interactions.

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WEAKLY SELF-AVOIDING WALK IN A PARETO-DISTRIBUTED RANDOM POTENTIAL

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We investigate a model of continuous-time simple random walk paths in \mathbb{Z}^d undergoing two competing interactions: an attractive one toward the large values of a random potential and a self-repellent one in the spirit of the well-known weakly self-avoiding random walk. We take the potential to be i.i.d. Pareto-distributed with parameter $\alpha > d$, and we tune the strength of the interactions in such a way that they both contribute on the same scale as $t \rightarrow \infty$.

Our main results are: (1) the identification of the logarithmic asymptotics of the partition function of the model in terms of a random variational formula and (2) the identification of the path behaviour that gives the overwhelming contribution to the partition function for $\alpha > 2d$: the random-walk path follows an optimal trajectory that visits each of a finite number of random lattice sites for a positive random fraction of time. We prove a law of large numbers for this behaviour, that is, that all other path behaviours give strictly less contribution to the partition function. The joint distribution of the variational problem and of the optimal path can be expressed in terms of a limiting Poisson point process arising by a rescaling of the random potential. The latter convergence is in distribution and is in the spirit of a standard extreme-value setting for a rescaling of an i.i.d. potential in large boxes, like in (*Ann. Probab.* **37** (2009) 347–392).

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QUENCHED LOCAL LIMIT THEOREM FOR RANDOM CONDUCTANCE MODELS WITH LONG-RANGE JUMPS

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We establish the quenched local limit theorem for reversible random walk on \mathbb{Z}^d (with $d \geq 2$) among stationary ergodic random conductances that permit jumps of arbitrary length. The proof is based on the weak parabolic Harnack inequalities and on-diagonal heat-kernel estimates for long-range random walks on general ergodic environments. In particular, this partly solves (*Probab. Theory Related Fields* **180** (2021) 847–889), Open Problem 2.7, where the quenched invariance principle was obtained. As a byproduct, we prove the maximal inequality with an extra tail term for long-range reversible random walks, which in turn yields the everywhere sublinear property for the associated corrector.

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THE MAKER–BREAKER PERCOLATION GAME ON A RANDOM BOARD

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The (m, b) Maker–Breaker percolation game on $(\mathbb{Z}^2)_p$, introduced by Day and Falgas-Ravry, is played in the following way. Before the game starts, each edge of \mathbb{Z}^2 is removed independently with probability $1 - p$. After that, Maker chooses a vertex v_0 to protect. Then in each round Maker and Breaker claim, respectively, m and b unclaimed edges of \mathbb{Z}^2 . Breaker wins if after the removal of the edges claimed by him the component of v_0 becomes finite, and Maker wins if she can indefinitely prevent Breaker from winning.

We show that for any $p < 1$, almost surely Breaker has a winning strategy for the $(1, 1)$ game on $(\mathbb{Z}^2)_p$. This fully answers a question of Day and Falgas-Ravry, who showed that for $p = 1$ Maker has a winning strategy for the $(1, 1)$ game. Further, we show that in the $(2, 1)$ game on $(\mathbb{Z}^2)_p$ almost surely Maker has a winning strategy whenever $p > 0.9402$, while almost surely Breaker has a winning strategy whenever $p < 0.5278$. This shows that the threshold value of p above which Maker has a winning strategy for the $(2, 1)$ game on \mathbb{Z}^2 is nontrivial. In fact, we prove similar results in various settings, including other lattices and biases (m, b) .

These results extend also to the most general case, which we introduce, where each edge is given to Maker with probability α and to Breaker with probability β before the game starts.

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GLOBAL WELL-POSEDNESS OF THE DYNAMICAL SINE-GORDON MODEL UP TO 6π

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We prove the global well-posedness of the dynamical sine-Gordon model up to the third threshold, that is, for parameters $\beta^2 < 6\pi$. The key novelty in our approach is the introduction of the so-called resonant equation, whose solution is entirely deterministic and completely captures the size of the solution to the dynamical sine-Gordon model. The probabilistic fluctuations in the dynamical sine-Gordon model are then controlled using uniform estimates for modified stochastic objects.

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IRREDUCIBLE VIRASORO REPRESENTATIONS IN LIOUVILLE CONFORMAL FIELD THEORY

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This paper studies the analytic continuation of Liouville eigenstates and shows that they assemble into irreducible highest-weight representations of the Virasoro algebra, for all values of the conformal weights. This builds on previous results from the first author and Guillarmou, Kupiainen, Rhodes & Vargas, where such representations were constructed except for the conformal weight on the Kac table. In order to extend these results to the degenerate weights, we find explicit analytic expressions for the Virasoro descendants and uncover the probabilistic meaning of the Kac table.

In the algebraic approach to conformal field theory, the irreducibility is a crucial property that must be satisfied by the representations in the spectrum and is usually taken as an axiom. Computationally, it leads to the celebrated null-vector (or BPZ) equations for correlation functions and conformal blocks, which are the cornerstone of the integrability of the theory.

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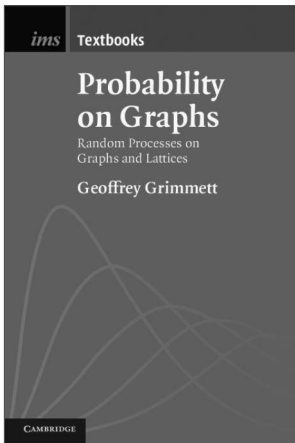
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