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MODERATE DEVIATIONS FOR THE CAPACITY OF THE RANDOM WALK RANGE IN DIMENSION FOUR

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In this paper we find a natural four-dimensional analog of the moderate deviation results for the capacity of the random walk, which corresponds to Bass, Chen and Rosen (*Mem. Amer. Math. Soc.* **198** (2009) viii+82) concerning the volume of the random walk range for $d = 2$. We find that the deviation statistics of the capacity of the random walk can be related to the following constant of generalized Gagliardo–Nirenberg inequalities:

$$\inf_{f:\|\nabla f\|_{L^2}<\infty} \frac{\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2}}{[\int_{(\mathbb{R}^4)^2} f^2(x)G(x-y)f^2(y)\,\mathrm{d}x\,\mathrm{d}y]^{1/4}}.$$

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ASYMPTOTIC PROPERTIES OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN THE SUBLINEAR REGIME

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In this paper we investigate stochastic heat equation with sublinear diffusion coefficients. By assuming certain concavity of the diffusion coefficient, we establish nontrivial moment upper bounds and almost sure spatial asymptotic properties for solutions. These results shed light on the *smoothing intermittency effect* under *weak diffusion* (i.e., sublinear growth) previously observed by Zeldovich et al. (*Proc. Natl. Acad. Sci. USA* **84** (1987) 6323–6325). The sample-path spatial asymptotics obtained in this paper partially bridge a gap in earlier works of Conus et al. (*Ann. Probab.* **41** (2013) 2225–2260; *Probab. Theory Related Fields* **156** (2013) 483–533), which focused on two extreme scenarios: a linear diffusion coefficient and a bounded diffusion coefficient. Our approach is highly robust and applicable to a variety of stochastic partial differential equations, including the one-dimensional stochastic wave equation.

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THE ⋆-VERTEX-REINFORCED JUMP PROCESS

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We investigate the nonreversible generalization of the Vertex-Reinforced Jump Process (VRJP), called the ⋆-Vertex-Reinforced Jump Process (⋆-VRJP) and introduced in Bacallado, Sabot, Tarrès (2020). It can be seen as the continuous-time counterpart to the ⋆-Edge-Reinforced Random Walk (⋆-ERRW) (see Bacallado, Chodera, Pande (2009); Bacallado, Sabot, Tarrès (2020)), which is itself a nonreversible (and in fact Yaglom reversible) generalization of the original ERRW introduced by Coppersmith and Diaconis in 1986 (Coppersmith and Diaconis (1986)). In contrast to the classical VRJP, the ⋆-VRJP is not exchangeable after time-change, which leads to several difficulties and new phenomena.

First, we show that with some appropriate randomization of the initial local time, it becomes partially exchangeable after time-change. We provide a representation of the "randomized" ⋆-VRJP as a mixture of Yaglom reversible Markov jump processes with an explicit mixing measure, and we prove that the nonrandomized ⋆-VRJP can be written as a mixture of conditioned Markov processes.

Second, we give a representation of the ⋆-VRJP in terms of a random Schrödinger operator. The corresponding representation for the classical VRJP has proved to be very useful in the understanding of its asymptotic behavior. The construction is based on several new and rather remarkable identities between integrals on the space of ⋆-symmetric and ⋆-antisymmetric functions on vertices. We give a description of the randomized ⋆-VRJP in terms of that random Schrödinger operator, which allows us to prove the representation of the randomized ⋆-VRJP as a mixture of Markov jump processes in a different and more analytic manner. Similar to the VRJP, we think that the representation by a random Schrödinger operator and the associated identities are key features of the ⋆-VRJP.

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ALMOST SHARP SHARPNESS FOR POISSON BOOLEAN PERCOLATION

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We consider Poisson Boolean percolation on $\mathbb{R}^d$ with power-law distribution on the radius with a finite d-moment for $d \geq 2$. We prove that subcritical sharpness occurs for all but a countable number of power-law distributions. This extends the results of Duminil-Copin–Raoufi–Tassion (*Ann. Henri Lebesgue* **3** (2020) 677–700) where subcritical sharpness is proved under the assumption that the radii distribution has a $5d-3$ finite moment. Our proofs techniques are different from (*Ann. Henri Lebesgue* **3** (2020) 677–700): we do not use a randomized algorithm and rely on specific independence properties of Boolean percolation, inherited from the underlying Poisson process.

We also prove supercritical sharpness for any distribution with a finite d-moment and the continuity of the critical parameter for the truncated distribution when the truncation goes to infinity.

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CURVATURE AND OTHER LOCAL INEQUALITIES IN MARKOV SEMIGROUPS

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We investigate the relationship between curvature bounds and local functional inequalities. In particular, we show that local inequalities derived through the maximum principle approach of Ivanisvili and Volberg (In *50 Years with Hardy Spaces* (2018) 281–305, Birkhäuser/Springer) are equivalent to lower bounds on the Ricci curvature in the sense of Bakry–Emery. Moreover, we will develop this Euclidean space technique to metric measure spaces satisfying the RCD condition, providing a unified approach to functional and isoperimetric inequalities in nonsmooth spaces with a synthetic Ricci curvature bound. Finally, we are interested in commutation properties for semigroup operators on $\mathbb{R}^n$ in the absence of positive curvature, based on a local eigenvalue criteria.

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SHARP CONVERGENCE RATES FOR MEAN FIELD CONTROL IN THE REGION OF STRONG REGULARITY

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We study the convergence problem for mean field control, also known as optimal control of McKean–Vlasov dynamics. We assume that the data is smooth but not convex, and thus the limiting value function $\mathcal{U} : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is Lipschitz, but may not be continuously differentiable. In this setting, the first and last named authors recently identified an open and dense subset $\mathcal{O}$ of $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ on which $\mathcal{U}$ is $\mathcal{C}^1$ and solves the relevant infinite-dimensional Hamilton–Jacobi equation in a classical sense. In the present paper we use these regularity results, and some nontrivial extensions of them, to derive sharp rates of convergence. In particular, we show that the value functions for the N-particle control problems converge toward $\mathcal{U}$ with a rate of $1/N$, uniformly on subsets of $\mathcal{O}$ which are compact in the p-Wasserstein space for some $p > 2$. A similar result is also established at the level of the optimal feedback controls. The rate $1/N$ is the optimal rate in this setting, even if $\mathcal{U}$ is smooth, while, in general, the optimal global rate of convergence is known to be slower than $1/N$. Thus, our results show that the rate of convergence is faster inside of $\mathcal{O}$ than it is outside. As a consequence of the convergence of the optimal feedbacks, we obtain a concentration inequality for optimal trajectories of the N-particle problem started from i.i.d. initial conditions.

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p-BROWNIAN MOTION AND THE p-LAPLACIAN

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In this paper we construct a stochastic process, more precisely, a (nonlinear) Markov process, which is related to the parabolic p-Laplace equation in the same way as Brownian motion is to the classical heat equation given by the (2-) Laplacian.

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FLUCTUATIONS OF THE HORTON–STRAHLER NUMBER OF STABLE GALTON–WATSON TREES

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The Horton–Strahler number—also called the register function—is a combinatorial tool that quantifies the branching complexity of a rooted tree. We study the law of the Horton–Strahler number of the canonical stable Galton–Watson trees conditioned to have size n (including the Catalan trees), which are the finite-dimensional marginals of stable Lévy trees. While these random variables are known to grow as a multiple of $\ln n$ in probability, their fluctuations are not well understood because they are coupled with deterministic oscillations. To rule out the latter, we introduce a real-valued variant of the Horton–Strahler number. We show that a rescaled exponential of this quantity jointly converges in distribution to a measurable function of the scaling limit of the trees, that is, the stable Lévy tree. We call this limit the Strahler dilation, and we discuss its similarities with the Horton–Strahler number.

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ON THE SOBOLEV REMOVABILITY OF THE GRAPH OF ONE-DIMENSIONAL BROWNIAN MOTION

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Suppose that B is a one-dimensional Brownian motion, and let $\Gamma = \{(t, B_t) : t \in [0, 1]\}$ be the graph of $B|_{[0,1]}$. We characterize the Sobolev removability properties of Γ by showing that Γ is almost surely not $W^{1,p}$-removable for all $p \in [1, \infty)$ but is almost surely $W^{1,\infty}$-removable.

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CUTOFF FOR THE GLAUBER-EXCLUSION PROCESS IN THE FULL HIGH-TEMPERATURE REGIME: AN INFORMATION PERCOLATION APPROACH

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The Glauber-Exclusion process is a superposition of a Glauber dynamics and the Symmetric Simple Exclusion Process (SSEP) on the lattice. The model was shown to admit a reaction-diffusion equation as the hydrodynamic limit. In this article we define a notion of temperature regimes via the reaction function in the equation and prove cutoff in the full high-temperature regime for the attractive model in dimensions 1 and 2 with periodic boundary condition. Our results show that the equation in the hydrodynamic limit reflects the mixing behavior of the large but finite system. Besides, cutoff is proved despite the lack of reversibility and an explicit formula for the invariant measure. We also provide the spectral gap and prove pre-cutoff in all dimensions. Our proof involves a new interpretation of attractiveness, the information percolation framework introduced by Lubetzky and Sly, anticoncentration of simple random walk on the lattice, and a coupling inspired by excursion theory. We hope that this approach can find new applications in the future.

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DIMENSION OF BERNOULLI CONVOLUTIONS IN $\mathbb{R}^d$

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For $(\lambda_1, \dots, \lambda_d) = \lambda \in (0,1)^d$ with $\lambda_1 > \cdots > \lambda_d$, denote by μ_λ the Bernoulli convolution associated to λ. That is, μ_λ is the distribution of the random vector $\sum_{n\ge 0} \pm(\lambda_1^n, \dots, \lambda_d^n)$, where the $\pm$ signs are chosen independently and with equal weight. Assuming for each $1 \le j \le d$ that λ_j is not a root of a polynomial with coefficients $\pm 1, 0$, we prove that the dimension of μ_λ equals $\min\{\dim_L \mu_\lambda, d\}$, where $\dim_L \mu_\lambda$ is the Lyapunov dimension. More generally, we obtain this result in the context of homogeneous diagonal self-affine systems on $\mathbb{R}^d$ with rational translations.

The proof extends to higher dimensions the works of Breuillard and Varjú and Varjú regarding Bernoulli convolutions on the real line. The main novelty and contribution of the present work lies in an extension of an entropy increase result, due to Varjú, in which the amount of increase in entropy is given explicitly. The extension of this result to the higher-dimensional nonconformal case requires significant new ideas.

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HEAT KERNEL ESTIMATES FOR DIRICHLET FORMS DEGENERATE AT THE BOUNDARY

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The goal of this paper is to establish sharp two-sided estimates on the heat kernels of two types of purely discontinuous symmetric Markov processes in the upper half-space of $\mathbb{R}^d$ with jump kernels degenerate at the boundary. The jump kernels are of the form $J(x, y) = \mathcal{B}(x, y)|x - y|^{-\alpha-d}$, $\alpha \in (0, 2)$, where the function $\mathcal{B}$ depends on four parameters and may vanish at the boundary. Our results are the first sharp two-sided estimates for the heat kernels of nonlocal operators with jump kernels degenerate at the boundary.

The first type of processes are conservative Markov processes on $\overline{\mathbb{R}}^d_+$ with jump kernel $J(x, y)$. Depending on the regions where the parameters belong, the heat kernels estimates have three different forms, two of them are qualitatively different from all previously known heat kernel estimates.

The second type of processes are the processes above killed either by a critical potential or upon hitting the boundary of the half-space. We establish that their heat kernel estimates have the approximate factorization property with survival probabilities decaying as a power of the distance to the boundary, where the power depends on the constant in the critical potential.

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UNIQUENESS OF GENERALIZED CONFORMAL RESTRICTION MEASURES AND MALLIAVIN–KONTSEVICH–SUHOV MEASURES FOR $c \in (0, 1]$

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In this paper, we present a unified approach to establish the uniqueness of generalized conformal restriction measures with central charge $c \in (0, 1]$ in both chordal and radial cases, by relating these measures to the Brownian loop soup. Our method also applies to the uniqueness of the Malliavin–Kontsevich–Suhov loop measures for $c \in (0, 1]$, which was recently obtained in (Baverez and Jego (2024)) for all $c \leq 1$ from a CFT framework of SLE loop measures. In contrast, though only valid for $c \in (0, 1]$, our approach provides additional probabilistic insights, as it directly links natural quantities of MKS measures to loop-soup observables.

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