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THE FYODOROV–HIARY–KEATING CONJECTURE ON MESOSCOPIC INTERVALS

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We derive precise upper bounds for the maximum of the Riemann zeta function on a typical short interval of the critical line. We show that for fixed $\theta \in (-1, 0]$, large T, and $y \geq 2$ satisfying $y = O(\log\log T/\log\log\log T)$, the proportion of points $t \in [T, 2T]$ for which

$$\max_{|h|\leq \log^\theta T} \left|\zeta\left(\tfrac{1}{2} + it + ih\right)\right| > e^y \cdot e^{S\sqrt{(\log\log T)|\theta|/2}} \frac{(\log T)^{(1+\theta)}}{(\log\log T)^{3/4}}$$

is bounded above by a constant times $y \exp(-2y - y^2/((1+\theta)\log\log T))$, where $S = S(t)$ is a quantity whose value distribution is approximately that of a standard Gaussian. Up to a multiplicative constant, this settles the upper bound of a conjecture of Fyodorov–Hiary–Keating which was only known in the leading order for $\theta \in (-1, 0)$.

Using similar techniques, we also derive upper bounds for the second moment of the zeta function on such intervals. We show that for large T, the proportion of $t \in [T, 2T]$ for which

$$\frac{1}{\log^\theta T} \int_{-\log^\theta T}^{\log^\theta T} \left|\zeta\left(\tfrac{1}{2} + it + ih\right)\right|^2 \mathrm{d}h > Ae^{S\sqrt{2|\theta|\log\log T}} \frac{(\log T)^{(1+\theta)}}{\sqrt{\log\log T}}$$

tends to zero as $A \to \infty$, for the same S as above. This proves a weak form of another conjecture of Fyodorov–Keating and generalizes a result of Harper, which is recovered at $\theta = 0$ (in which case S is defined to be zero). Our proofs use an adaptation of the recursive scheme introduced by one of the authors, Bourgade and Radziwiłł in (Arguin, Bourgade and Radziwiłł (2020)).

REFERENCES

[1] ARGUIN, L.-P. and BAILEY, E. (2023). Large deviation estimates of Selberg's central limit theorem and applications. *Int. Math. Res. Not. IMRN* **2023** No. 23. 20574–20612. MR4675078 https://doi.org/10.1093/imrn/rnad176
[2] ARGUIN, L.-P. and BAILEY, E. (2025). Lower bounds for the large deviations of Selberg's central limit theorem. *Mathematika* **71** e70002. MR4838549 https://doi.org/10.1112/mtk.70002
[3] ARGUIN, L.-P., BELIUS, D. and BOURGADE, P. (2017). Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.* **349** 703–751. MR3594368 https://doi.org/10.1007/s00220-016-2740-6
[4] ARGUIN, L.-P., BELIUS, D., BOURGADE, P., RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2019). Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure Appl. Math.* **72** 500–535. MR3911893 https://doi.org/10.1002/cpa.21791
[5] ARGUIN, L.-P., BELIUS, D. and HARPER, A. J. (2017). Maxima of a randomized Riemann zeta function, and branching random walks. *Ann. Appl. Probab.* **27** 178–215. MR3619786 https://doi.org/10.1214/16-AAP1201
[6] ARGUIN, L.-P., BOURGADE, P. and RADZIWIŁŁ, M. (2020). The Fyodorov–Hiary–Keating conjecture. I. Available at arXiv:2007.00988.
[7] ARGUIN, L.-P., BOURGADE, P. and RADZIWIŁŁ, M. (2023). The Fyodorov–Hiary–Keating conjecture. II. Available at arXiv:2307.00982.

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[8] ARGUIN, L.-P. and CREIGHTON, N. (2026). Lower bounds for the large deviations and moments of the Riemann zeta function on the critical line. Available at arXiv:2603.01711.

[9] ARGUIN, L.-P., DUBACH, G. and HARTUNG, L. (2024). Maxima of a random model of the Riemann zeta function over intervals of varying length. *Ann. Inst. Henri Poincaré Probab. Stat.* **60** 588–611. MR4718391 https://doi.org/10.1214/22-aihp1323

[10] ARGUIN, L.-P., OUIMET, F. and RADZIWIŁŁ, M. (2021). Moments of the Riemann zeta function on short intervals of the critical line. *Ann. Probab.* **49** 3106–3141. MR4348686 https://doi.org/10.1214/21-aop1524

[11] BAILEY, E. C. and KEATING, J. P. (2022). Maxima of log-correlated fields: Some recent developments. *J. Phys. A* **55** 053001. MR4407477 https://doi.org/10.1088/1751-8121/ac4394

[12] BOMBIERI, E. and FRIEDLANDER, J. B. (1995). Dirichlet polynomial approximations to zeta functions. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (4) **22** 517–544. MR1360548

[13] BOURGADE, P. (2010). Mesoscopic fluctuations of the zeta zeros. *Probab. Theory Related Fields* **148** 479–500. MR2678896 https://doi.org/10.1007/s00440-009-0237-3

[14] BOVIER, A. and HARTUNG, L. (2020). From 1 to 6: A finer analysis of perturbed branching Brownian motion. *Comm. Pure Appl. Math.* **73** 1490–1525. MR4156608 https://doi.org/10.1002/cpa.21893

[15] BRAMSON, M. D. (1978). Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31** 531–581. No. 5. MR494541. https://doi.org/10.1002/cpa.3160310502

[16] CHANG, C. (2024). Hybrid statistics of a random model of zeta over intervals of varying length.

[17] CHHAIBI, R., MADAULE, T. and NAJNUDEL, J. (2018). On the maximum of the CβE field. *Duke Math. J.* **167** 2243–2345. MR3848391 https://doi.org/10.1215/00127094-2018-0016

[18] DIACONIS, P. and EVANS, S. N. (2001). Linear functionals of eigenvalues of random matrices. *Trans. Amer. Math. Soc.* **353** 2615–2633. MR1828463 https://doi.org/10.1090/S0002-9947-01-02800-8

[19] FYODOROV, Y. V., HIARY, G. A. and KEATING, J. P. (2012). Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Phys. Rev. Lett.* **108** No. 17. 170601.

[20] FYODOROV, Y. V. and KEATING, J. P. (2014). Freezing transitions and extreme values: Random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **372** 20120503. MR3151088 https://doi.org/10.1098/rsta.2012.0503

[21] HARDY, G. H. and LITTLEWOOD, J. E. (1916). Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. *Acta Math.* **41** 119–196. MR1555148 https://doi.org/10.1007/BF02422942

[22] HARPER, A. J. (2018-2019). The Riemann zeta function in short intervals [after Najnudel, and Arguin, Belius, Bourgade, Radziwiłł, and Soundararajan]. *Sémin. Bourb.* 71*e Ann.* **1159**.

[23] HARPER, A. J. (2019). On the partition function of the Riemann zeta function, and the Fyodorov–Hiary–Keating conjecture. Available at arXiv:1906.05783.

[24] IVIĆ, A. (1985). *The Riemann Zeta-Function. A Wiley-Interscience Publication*. Wiley, New York. The theory of the Riemann zeta-function with applications. MR0792089

[25] JUNNILA, J., LAMBERT, G. and WEBB, C. (2024). Multiplicative chaos measures from thick points of log-correlated fields. *Comm. Pure Appl. Math.* **77** 4212–4286. MR4814919 https://doi.org/10.1002/cpa.22205

[26] KEATING, J. P. and SNAITH, N. C. (2000). Random matrix theory and $\zeta(1/2+it)$. *Comm. Math. Phys.* **214** 57–89. No. 1. MR1794265. https://doi.org/10.1007/s002200000261

[27] KISTLER, N. and SCHMIDT, M. A. (2015). From Derrida's random energy model to branching random walks: From 1 to 3. *Electron. Commun. Probab.* **20** 47. MR3358969 https://doi.org/10.1214/ecp.v20-4189

[28] KOUKOULOPOULOS, D. (2019). *The distribution of prime numbers*. *Graduate Studies in Mathematics* **203**. American Mathematical Society, Providence, RI. xii + 356 pp. ISBN 978-1-4704-4754-0; 978-1-4704-6285-7. MR3971232. https://doi.org/10.1090/gsm/203

[29] MONTGOMERY, H. L. and VAUGHAN, R. C. (1974). Hilbert's inequality. *J. Lond. Math. Soc.* (2) **8** 73–82. MR0337775 https://doi.org/10.1112/jlms/s2-8.1.73

[30] MONTGOMERY, H. L. and VAUGHAN, R. C. (2007). *Multiplicative Number Theory. I. Classical Theory*. *Cambridge Studies in Advanced Mathematics* **97**. Cambridge Univ. Press, Cambridge. MR2378655

[31] NAJNUDEL, J. (2018). On the extreme values of the Riemann zeta function on random intervals of the critical line. *Probab. Theory Related Fields* **172** 387–452. MR3851835 https://doi.org/10.1007/s00440-017-0812-y

[32] NIKULA, M., SAKSMAN, E. and WEBB, C. (2020). Multiplicative chaos and the characteristic polynomial of the CUE: The L^1-phase. *Trans. Amer. Math. Soc.* **373** 3905–3965. MR4105514 https://doi.org/10.1090/tran/8020

[33] PAQUETTE, E. and ZEITOUNI, O. (2018). The maximum of the CUE field. *Int. Math. Res. Not. IMRN* **2018** No. 16. 5028–5119. MR3848227 https://doi.org/10.1093/imrn/rnx033

[34] PAQUETTE, E. and ZEITOUNI, O. (2025). The extremal landscape for the CβE ensemble. *Forum Math. Sigma* **13** e1. MR4849370 https://doi.org/10.1017/fms.2024.129

[35] POWELL, E. (2021). Critical Gaussian multiplicative chaos: a review. *Markov Process. Related Fields* **27** 506–557. No. 4. MR4396197.

[36] RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2017). Selberg's central limit theorem for $\log|\zeta(1/2+it)|$. *Enseign. Math.* **63** 1–19. MR3832861 https://doi.org/10.4171/LEM/63-1/2-1

[37] REMY, G. (2020). The Fyodorov–Bouchaud formula and Liouville conformal field theory. *Duke Math. J.* **169** 177–211. MR4047550 https://doi.org/10.1215/00127094-2019-0045

[38] SAKSMAN, E. and WEBB, C. (2020). The Riemann zeta function and Gaussian multiplicative chaos: Statistics on the critical line. *Ann. Probab.* **48** 2680–2754. MR4164452 https://doi.org/10.1214/20-AOP1433

[39] SELBERG, A. (1946). Contributions to the theory of the Riemann zeta-function. *Arch. Math. Naturvidensk.* **48** 89–155. MR0020594

[40] SELBERG, A. (1992). Old and new conjectures and results about a class of Dirichlet series. In *Proceedings of the Amalfi Conference on Analytic Number Theory* (*Maiori*, 1989) 367–385. Univ. Salerno, Salerno. MR1220477

[41] SOUNDARARAJAN, K. (2009). Moments of the Riemann zeta function. *Ann. of Math.* (2) **170** 981–993. MR2552116 https://doi.org/10.4007/annals.2009.170.981

[42] SUBAG, E. and ZEITOUNI, O. (2015). Freezing and decorated Poisson point processes. *Comm. Math. Phys.* **337** 55–92. MR3324155 https://doi.org/10.1007/s00220-015-2303-2

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EXISTENCE OF BASS MARTINGALES AND THE MARTINGALE BENAMOU–BRENIER PROBLEM IN $\mathbb{R}^d$

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In classical optimal transport, the contributions of Benamou–Brenier and McCann regarding the time-dependent version of the problem are cornerstones of the field and form the basis for a variety of applications in other mathematical areas.

In this article we characterize solutions to the martingale Benamou–Brenier problem as *Bass martingales*, that is, transformations of Brownian motion through the gradient of a convex function. Our result is based on a new (static) Brenier-type theorem for a particular weak martingale optimal transport problem. As in the classical case, the structure of the primal optimizer is derived from its dual counterpart, whose derivation forms the technical core of this article. A key challenge is that dual attainment is a subtle issue in martingale optimal transport, where dual optimizers may fail to exist, even in highly regular settings.

REFERENCES

[1] ACCIAIO, B., MARINI, A. and PAMMER, G. (2025). Calibration of the Bass local volatility model. *SIAM J. Financial Math.* **16** 803–833. MR4941918 https://doi.org/10.1137/23M1622660

[2] ACCIAIO, B. and PAMMER, G. (2022). A short proof of the characterisation of convex order using the 2-Wasserstein distance. Available at arXiv:2207.02096.

[3] AMBROSIO, L. and GIGLI, N. (2013). A user's guide to optimal transport. In *Modelling and Optimisation of Flows on Networks*. *Lecture Notes in Math.* **2062** 1–155. Springer, Heidelberg. MR3050280 https://doi.org/10.1007/978-3-642-32160-3_1

[4] BACKHOFF-VERAGUAS, J., BARTL, D., BEIGLBÖCK, M. and EDER, M. (2020). Adapted Wasserstein distances and stability in mathematical finance. *Finance Stoch.* **24** 601–632. MR4118990 https://doi.org/10.1007/s00780-020-00426-3

[5] BACKHOFF-VERAGUAS, J., BEIGLBÖCK, M., HUESMANN, M. and KÄLLBLAD, S. (2020). Martingale Benamou–Brenier: A probabilistic perspective. *Ann. Probab.* **48** 2258–2289. MR4152642 https://doi.org/10.1214/20-AOP1422

[6] BACKHOFF-VERAGUAS, J., BEIGLBÖCK, M. and PAMMER, G. (2019). Existence, duality, and cyclical monotonicity for weak transport costs. *Calc. Var. Partial Differential Equations* **58** Paper No. 203. MR4029731 https://doi.org/10.1007/s00526-019-1624-y

[7] BASS, R. F. (1983). Skorokhod imbedding via stochastic integrals. In *Seminar on Probability, XVII* (J. Azéma and M. Yor, eds.). *Lecture Notes in Math.* **986** 221–224. Springer, Berlin. MR0770414 https://doi.org/10.1007/BFb0068318

[8] BAUSCHKE, H. H. and COMBETTES, P. L. (2017). *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 2nd ed. *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC*. Springer, Cham. MR3616647 https://doi.org/10.1007/978-3-319-48311-5

[9] BEIGLBÖCK, M., COX, A. M. G. and HUESMANN, M. (2017). Optimal transport and Skorokhod embedding. *Invent. Math.* **208** 327–400. MR3639595 https://doi.org/10.1007/s00222-016-0692-2

[10] BEIGLBÖCK, M., HENRY-LABORDÈRE, P. and PENKNER, F. (2013). Model-independent bounds for option prices—a mass transport approach. *Finance Stoch.* **17** 477–501. MR3066985 https://doi.org/10.1007/s00780-013-0205-8

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[11] BEIGLBÖCK, M., HENRY-LABORDÈRE, P. and TOUZI, N. (2017). Monotone martingale transport plans and Skorokhod embedding. *Stoch. Process. Appl.* **127** 3005–3013. MR3682121 https://doi.org/10.1016/j.spa.2017.01.004

[12] BEIGLBÖCK, M. and JUILLET, N. (2016). On a problem of optimal transport under marginal martingale constraints. *Ann. Probab.* **44** 42–106. MR3456332 https://doi.org/10.1214/14-AOP966

[13] BEIGLBÖCK, M., LIM, T. and OBŁÓJ, J. (2019). Dual attainment for the martingale transport problem. *Bernoulli* **25** 1640–1658. MR3961225 https://doi.org/10.3150/17-BEJ1015

[14] BEIGLBÖCK, M., NUTZ, M. and STEBEGG, F. (2022). Fine properties of the optimal Skorokhod embedding problem. *J. Eur. Math. Soc.* (*JEMS*) **24** 1389–1429. MR4397044 https://doi.org/10.4171/JEMS/1122

[15] BEIGLBÖCK, M., NUTZ, M. and TOUZI, N. (2017). Complete duality for martingale optimal transport on the line. *Ann. Probab.* **45** 3038–3074. MR3706738 https://doi.org/10.1214/16-AOP1131

[16] BEIGLBÖCK, M., PAMMER, G., RIESS, L. and SCHROTT, S. (2025). The fundamental theorem of weak optimal transport. Available at arXiv:2501.16316.

[17] BENAMOU, J.-D. and BRENIER, Y. (1999). A numerical method for the optimal time-continuous mass transport problem and related problems. In *Monge Ampère Equation*: *Applications to Geometry and Optimization* (*Deerfield Beach*, *FL*, 1997). *Contemp. Math.* **226** 1–11. Amer. Math. Soc., Providence, RI. MR1660739 https://doi.org/10.1090/conm/226/03232

[18] BOUCHARD, B. and NUTZ, M. (2015). Arbitrage and duality in nondominated discrete-time models. *Ann. Appl. Probab.* **25** 823–859. MR3313756 https://doi.org/10.1214/14-AAP1011

[19] BRENIER, Y. (1987). Décomposition polaire et réarrangement monotone des champs de vecteurs. *C. R. Acad. Sci. Paris Sér. I Math.* **305** 805–808. MR0923203

[20] BRENIER, Y. (1991). Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.* **44** 375–417. MR1100809 https://doi.org/10.1002/cpa.3160440402

[21] CAMPI, L., LAACHIR, I. and MARTINI, C. (2017). Change of numeraire in the two-marginals martingale transport problem. *Finance Stoch.* **21** 471–486. MR3626622 https://doi.org/10.1007/s00780-016-0322-2

[22] CHERIDITO, P., KIISKI, M., PRÖMEL, D. J. and SONER, H. M. (2021). Martingale optimal transport duality. *Math. Ann.* **379** 1685–1712. MR4238277 https://doi.org/10.1007/s00208-019-01952-y

[23] CIOSMAK, K. (2024). Localisation for constrained transports I: Theory. Available at arXiv:2312.12281.

[24] CIOSMAK, K. (2024). Localisation for constrained transports II: Applications. Available at arXiv:2312.13167.

[25] CONZE, A. and HENRY-LABORDERE, P. (2021). Bass construction with multi-marginals: Lightspeed computation in a new local volatility model. *SSRN Electron. J.*.

[26] COX, A. M. G., OBŁÓJ, J. and TOUZI, N. (2019). The Root solution to the multi-marginal embedding problem: An optimal stopping and time-reversal approach. *Probab. Theory Related Fields* **173** 211–259. MR3916107 https://doi.org/10.1007/s00440-018-0833-1

[27] DE MARCH, H. and TOUZI, N. (2019). Irreducible convex paving for decomposition of multidimensional martingale transport plans. *Ann. Probab.* **47** 1726–1774. MR3945758 https://doi.org/10.1214/18-AOP1295

[28] DOLINSKY, Y. and SONER, H. M. (2014). Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Related Fields* **160** 391–427. MR3256817 https://doi.org/10.1007/s00440-013-0531-y

[29] FÖLLMER, H. (2022). Optimal couplings on Wiener space and an extension of Talagrand's transport inequality. In *Stochastic Analysis*, *Filtering*, *and Stochastic Optimization* 147–175. Springer, Cham. MR4433813 https://doi.org/10.1007/978-3-030-98519-6_7

[30] GALICHON, A., HENRY-LABORDÈRE, P. and TOUZI, N. (2014). A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.* **24** 312–336. MR3161649 https://doi.org/10.1214/13-AAP925

[31] GHOUSSOUB, N., KIM, Y.-H. and LIM, T. (2019). Structure of optimal martingale transport plans in general dimensions. *Ann. Probab.* **47** 109–164. MR3909967 https://doi.org/10.1214/18-AOP1258

[32] GHOUSSOUB, N., KIM, Y.-H. and LIM, T. (2020). Optimal Brownian stopping when the source and target are radially symmetric distributions. *SIAM J. Control Optim.* **58** 2765–2789. MR4149541 https://doi.org/10.1137/19M1270513

[33] GHOUSSOUB, N., KIM, Y.-H. and PALMER, A. Z. (2019). PDE methods for optimal Skorokhod embeddings. *Calc. Var. Partial Differential Equations* **58** Paper No. 113. MR3959935 https://doi.org/10.1007/s00526-019-1563-7

[34] GOZLAN, N. and JUILLET, N. (2020). On a mixture of Brenier and Strassen theorems. *Proc. Lond. Math. Soc.* (3) **120** 434–463. MR4008375 https://doi.org/10.1112/plms.12302

[35] GOZLAN, N., ROBERTO, C., SAMSON, P.-M. and TETALI, P. (2017). Kantorovich duality for general transport costs and applications. *J. Funct. Anal.* **273** 3327–3405. MR3706606 https://doi.org/10.1016/j.jfa.2017.08.015

[36] GUO, I. and LOEPER, G. (2021). Path dependent optimal transport and model calibration on exotic derivatives. *Ann. Appl. Probab.* **31** 1232–1263. MR4278783 https://doi.org/10.1214/20-aap1617

[37] GUO, I., LOEPER, G. and WANG, S. (2019). Local volatility calibration by optimal transport. In 2017 *MATRIX Annals. MATRIX Book Ser.* **2** 51–64. Springer, Cham. MR3931598

[38] HENRY-LABORDÈRE, P., OBŁÓJ, J., SPOIDA, P. and TOUZI, N. (2016). The maximum maximum of a martingale with given n marginals. *Ann. Appl. Probab.* **26** 1–44. MR3449312 https://doi.org/10.1214/14-AAP1084

[39] HENRY-LABORDÈRE, P. and TOUZI, N. (2016). An explicit martingale version of the one-dimensional Brenier theorem. *Finance Stoch.* **20** 635–668. MR3519164 https://doi.org/10.1007/s00780-016-0299-x

[40] HOBSON, D. and NEUBERGER, A. (2012). Robust bounds for forward start options. *Math. Finance* **22** 31–56. MR2881879 https://doi.org/10.1111/j.1467-9965.2010.00473.x

[41] HUESMANN, M. and TREVISAN, D. (2019). A Benamou–Brenier formulation of martingale optimal transport. *Bernoulli* **25** 2729–2757. MR4003563 https://doi.org/10.3150/18-BEJ1069

[42] JOSEPH, B., LOEPER, G. and OBLOJ, J. (2024). The measure preserving martingale Sinkhorn algorithm. Available at arXiv:2310.13797.

[43] KÄLLBLAD, S., TAN, X. and TOUZI, N. (2017). Optimal Skorokhod embedding given full marginals and Azéma-Yor peacocks. *Ann. Appl. Probab.* **27** 686–719. MR3655851 https://doi.org/10.1214/16-AAP1191

[44] KANTOROVITCH, L. (1942). On the translocation of masses. *C. R. (Dokl.) Acad. Sci. URSS* **37** 199–201. MR0009619

[45] KOMLÓS, J. (1967). A generalization of a problem of Steinhaus. *Acta Math. Acad. Sci. Hung.* **18** 217–229. MR0210177 https://doi.org/10.1007/BF02020976

[46] LOEPER, G. (2018). Option pricing with linear market impact and nonlinear Black–Scholes equations. *Ann. Appl. Probab.* **28** 2664–2726. MR3847970 https://doi.org/10.1214/17-AAP1367

[47] MCCANN, R. J. (1994). A Convexity theory for interacting gases and equilibrum crystals. PhD thesis, Princeton University.

[48] MCCANN, R. J. (1995). Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.* **80** 309–323. MR1369395 https://doi.org/10.1215/S0012-7094-95-08013-2

[49] MCCANN, R. J. (2001). Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.* **11** 589–608. MR1844080 https://doi.org/10.1007/PL00001679

[50] MIRSKY, L. (1961). Majorization of vectors and inequalities for convex functions. *Monatsh. Math.* **65** 159–169. MR0123661 https://doi.org/10.1007/BF01307025

[51] MONGE, G. (1781). Mémoire sur la théorie des déblais et des remblais. *Hist. Acad. R. Sci. Mém. Math. Phys.*.

[52] OBŁÓJ, J. and SIORPAES, P. (2017). Structure of martingale transports in finite dimensions. Available at arXiv:1702.08433.

[53] OBŁÓJ, J., SPOIDA, P. and TOUZI, N. (2015). Martingale inequalities for the maximum via pathwise arguments. In *In Memoriam Marc Yor—Séminaire de Probabilités XLVII. Lecture Notes in Math.* **2137** 227–247. Springer, Cham. MR3444301 https://doi.org/10.1007/978-3-319-18585-9_11

[54] ROCKAFELLAR, R. T. (1997). *Convex Analysis. Princeton Landmarks in Mathematics*. Princeton Univ. Press, Princeton, NJ. MR1451876

[55] SANTAMBROGIO, F. (2015). *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling. Progress in Nonlinear Differential Equations and Their Applications* **87**. Birkhäuser/Springer, Cham. MR3409718 https://doi.org/10.1007/978-3-319-20828-2

[56] SCHACHERMAYER, W. and TSCHIDERER, B. (2024). The decomposition of stretched Brownian motion into Bass martingales. Available at arXiv:2406.10656.

[57] TAN, X. and TOUZI, N. (2013). Optimal transportation under controlled stochastic dynamics. *Ann. Probab.* **41** 3201–3240. MR3127880 https://doi.org/10.1214/12-AOP797

[58] VILLANI, C. (2003). *Topics in Optimal Transportation. Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483 https://doi.org/10.1090/gsm/058

[59] VILLANI, C. (2009). *Optimal Transport: Old and New. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* **338**. Springer, Berlin. MR2459454 https://doi.org/10.1007/978-3-540-71050-9

[60] WIESEL, J. and ZHANG, E. (2023). An optimal transport-based characterization of convex order. *Depend. Model.* **11** Paper No 20230102. MR4656034 https://doi.org/10.1515/demo-2023-0102

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WASSERSTEIN MIRROR GRADIENT FLOW AS THE LIMIT OF THE SINKHORN ALGORITHM

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We prove that the sequence of marginals obtained from the iterations of the Sinkhorn algorithm or the iterative proportional fitting procedure (IPFP) on joint densities, converges to an absolutely continuous curve on the 2-Wasserstein space, as the regularization parameter ε goes to zero and the number of iterations is scaled as $1/\varepsilon$ (and other technical assumptions). This limit, which we call the Sinkhorn flow, is an example of a Wasserstein mirror gradient flow, a concept we introduce here inspired by the well-known Euclidean mirror gradient flows. In the case of Sinkhorn, the gradient is that of the relative entropy functional with respect to one of the marginals, and the mirror is half of the squared Wasserstein distance functional from the other marginal. Interestingly, the norm of the velocity field of this flow can be interpreted as the metric derivative with respect to the linearized optimal transport (LOT) distance. An equivalent description of this flow is provided by the parabolic Monge–Ampère PDE whose connection to the Sinkhorn algorithm was noticed by Berman (*Numer. Math.* **145** (2020) 771–836). We derive conditions for exponential convergence for this limiting flow. We also construct a McKean–Vlasov diffusion whose marginal distributions follow the Sinkhorn flow.

REFERENCES

[1] ABEDIN, F. and KITAGAWA, J. (2020). Exponential convergence of parabolic optimal transport on bounded domains. *Anal. PDE* **13** 2183–2204. MR4175823 https://doi.org/10.2140/apde.2020.13.2183
[2] AHN, K. and CHEWI, S. (2021). Efficient constrained sampling via the mirror-Langevin algorithm. *Adv. Neural Inf. Process. Syst.* **34** 28405–28418.
[3] AMBROSIO, L., GIGLI, N. and SAVARÉ, G. (2008). *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed. *Lectures in Mathematics ETH Zürich*. Birkhäuser, Basel. MR2401600
[4] ARJOVSKY, M., CHINTALA, S. and BOTTOU, L. (2017). Wasserstein generative adversarial networks. In *International Conference on Machine Learning* 214–223. PMLR.
[5] ARNOLD, A., MARKOWICH, P., TOSCANI, G. and UNTERREITER, A. (2001). On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations* **26** 43–100. MR1842428 https://doi.org/10.1081/PDE-100002246
[6] BAKRY, D. and ÉMERY, M. (1985). Diffusions hypercontractives. In *Séminaire de Probabilités, XIX*, 1983/84. *Lecture Notes in Math.* **1123** 177–206. Springer, Berlin. MR0889476 https://doi.org/10.1007/BFb0075847
[7] BERMAN, R. J. (2020). The Sinkhorn algorithm, parabolic optimal transport and geometric Monge–Ampère equations. *Numer. Math.* **145** 771–836. MR4125978 https://doi.org/10.1007/s00211-020-01127-x
[8] BERNTON, E., GHOSAL, P. and NUTZ, M. (2022). Entropic optimal transport: Geometry and large deviations. *Duke Math. J.* **171** 3363–3400. MR4505361 https://doi.org/10.1215/00127094-2022-0035
[9] BHATTACHARYA, R. and WAYMIRE, E. C. (2023). *Continuous Parameter Markov Processes and Stochastic Differential Equations*. *Graduate Texts in Mathematics* **299**. Springer, Cham. MR4703938 https://doi.org/10.1007/978-3-031-33296-8

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[10] BOYD, S. and VANDENBERGHE, L. (2004). *Convex Optimization*. Cambridge Univ. Press, Cambridge. MR2061575 https://doi.org/10.1017/CBO9780511804441

[11] CAI, T., CHENG, J., CRAIG, N. and CRAIG, K. (2020). Linearized optimal transport for collider events. *Phys. Rev. D* **102** 116019.

[12] CARLIER, G. and LABORDE, M. (2020). A differential approach to the multi-marginal Schrödinger system. *SIAM J. Math. Anal.* **52** 709–717. MR4062805 https://doi.org/10.1137/19M1253800

[13] CARLIER, G., PEGON, P. and TAMANINI, L. (2023). Convergence rate of general entropic optimal transport costs. *Calc. Var. Partial Differential Equations* **62** Paper No. 116, 28 pp. MR4565039 https://doi.org/10.1007/s00526-023-02455-0

[14] CATTIAUX, P., GUILLIN, A. and WU, L.-M. (2010). A note on Talagrand's transportation inequality and logarithmic Sobolev inequality. *Probab. Theory Related Fields* **148** 285–304. MR2653230 https://doi.org/10.1007/s00440-009-0231-9

[15] CHAINTRON, L.-P. and DIEZ, A. (2022). Propagation of chaos: A review of models, methods and applications. II. Applications. *Kinet. Relat. Models* **15** 1017–1173. MR4489769 https://doi.org/10.3934/krm.2022018

[16] CHEN, H.-B., CHEWI, S. and NILES-WEED, J. (2021). Dimension-free log-Sobolev inequalities for mixture distributions. *J. Funct. Anal.* **281** Paper No. 109236, 17 pp. MR4316725 https://doi.org/10.1016/j.jfa.2021.109236

[17] CHEN, Y., GEORGIOU, T. and PAVON, M. (2016). Entropic and displacement interpolation: A computational approach using the Hilbert metric. *SIAM J. Appl. Math.* **76** 2375–2396. MR3579702 https://doi.org/10.1137/16M1061382

[18] CHEWI, S., LE GOUIC, T., LU, C., MAUNU, T., RIGOLLET, P. and STROMME, A. (2020). Exponential ergodicity of mirror-Langevin diffusions. *Adv. Neural Inf. Process. Syst.* **33** 19573–19585.

[19] CHIARINI, A., CONFORTI, G., GRECO, G. and TAMANINI, L. (2023). Gradient estimates for the Schrödinger potentials: Convergence to the Brenier map and quantitative stability. *Comm. Partial Differential Equations* **48** 895–943. MR4643676 https://doi.org/10.1080/03605302.2023.2215527

[20] CHIARINI, A., CONFORTI, G., GRECO, G. and TAMANINI, L. (2024). A semiconcavity approach to stability of entropic plans and exponential convergence of Sinkhorn's algorithm. Preprint. Available at arXiv:2412.09235.

[21] CHIZAT, L., ZHANG, S., HEITZ, M. and SCHIEBINGER, G. (2022). Trajectory inference via mean-field Langevin in path space. *Adv. Neural Inf. Process. Syst.* **35** 16731–16742.

[22] COMINETTI, R. and SAN MARTÍN, J. (1994). Asymptotic analysis of the exponential penalty trajectory in linear programming. *Math. Program.* **67** 169–187. MR1305565 https://doi.org/10.1007/BF01582220

[23] CONFORTI, G. and TAMANINI, L. (2021). A formula for the time derivative of the entropic cost and applications. *J. Funct. Anal.* **280** Paper No. 108964, 48 pp. MR4232667 https://doi.org/10.1016/j.jfa.2021.108964

[24] CSISZÁR, I. (1975). I-divergence geometry of probability distributions and minimization problems. *Ann. Probab.* **3** 146–158. MR0365798 https://doi.org/10.1214/aop/1176996454

[25] CUTURI, M. (2013). Sinkhorn distances: Lightspeed computation of optimal transport. *Adv. Neural Inf. Process. Syst.* **26**.

[26] DEB, N. and LIANG, T. (2025). No-regret generative modeling via parabolic Monge–Ampère PDE. Preprint. Available at arXiv:2504.09279.

[27] DELIGIANNIDIS, G., DE BORTOLI, V. and DOUCET, A. (2024). Quantitative uniform stability of the iterative proportional fitting procedure. *Ann. Appl. Probab.* **34** 501–516. MR4696283 https://doi.org/10.1214/23-aap1970

[28] DOBRUSHIN, L. (1979). Vlasov equations. *Funct. Anal. Appl.* **13** 115–123.

[29] DURMUS, A., MAJEWSKI, S. and MIASOJEDOW, B. (2019). Analysis of Langevin Monte Carlo via convex optimization. *J. Mach. Learn. Res.* **20** Paper No. 73, 46 pp. MR3960927

[30] DURRETT, R. (2019). *Probability—Theory and Examples*. *Cambridge Series in Statistical and Probabilistic Mathematics* **49**. Cambridge Univ. Press, Cambridge. MR3930614 https://doi.org/10.1017/9781108591034

[31] ECKSTEIN, S. and NUTZ, M. (2022). Quantitative stability of regularized optimal transport and convergence of Sinkhorn's algorithm. *SIAM J. Math. Anal.* **54** 5922–5948. MR4506579 https://doi.org/10.1137/21M145505X

[32] ECKSTEIN, S. and NUTZ, M. (2024). Convergence rates for regularized optimal transport via quantization. *Math. Oper. Res.* **49** 1223–1240. MR4755769 https://doi.org/10.1287/moor.2022.0245

[33] FRANKLIN, J. and LORENZ, J. (1989). On the scaling of multidimensional matrices. *Linear Algebra Appl.* **114/115** 717–735. MR0986904 https://doi.org/10.1016/0024-3795(89)90490-4

[34] GHOSAL, P. and NUTZ, M. (2022). On the convergence rate of Sinkhorn's algorithm. Preprint. Available at arXiv:2212.06000.

[35] GHOSAL, P., NUTZ, M. and BERNTON, E. (2022). Stability of entropic optimal transport and Schrödinger bridges. *J. Funct. Anal.* **283** Paper No. 109622, 22 pp. MR4460341 https://doi.org/10.1016/j.jfa.2022.109622

[36] GIGLI, N. and TAMANINI, L. (2018). Second order differentiation formula on RCD(K, N) spaces. *Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl.* **29** 377–386. MR3797990 https://doi.org/10.4171/RLM/811

[37] HOLLEY, R. and STROOCK, D. (1987). Logarithmic Sobolev inequalities and stochastic Ising models. *J. Stat. Phys.* **46** 1159–1194. MR0893137 https://doi.org/10.1007/BF01011161

[38] HORN, R. A. and JOHNSON, C. R. (1994). *Topics in Matrix Analysis*. Cambridge Univ. Press, Cambridge. MR1288752

[39] JABIN, P.-E. (2014). A review of the mean field limits for Vlasov equations. *Kinet. Relat. Models* **7** 661–711. MR3317577 https://doi.org/10.3934/krm.2014.7.661

[40] JORDAN, R., KINDERLEHRER, D. and OTTO, F. (1998). The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.* **29** 1–17. MR1617171 https://doi.org/10.1137/S0036141096303359

[41] KAC, M. (1956). Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1954–1955, *Vol. III* 171–197. Univ. California Press, Berkeley, CA. MR0084985

[42] KANTOROVITCH, L. (1942). On the translocation of masses. *C. R. (Dokl.) Acad. Sci. URSS* **37** 199–201. MR0009619

[43] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940 https://doi.org/10.1007/978-1-4612-0949-2

[44] KIM, Y.-H., STREETS, J. and WARREN, M. (2012). Parabolic optimal transport equations on manifolds. *Int. Math. Res. Not. IMRN* **2012** 4325–4350. MR2981711 https://doi.org/10.1093/imrn/rnr188

[45] KITAGAWA, J. (2012). A parabolic flow toward solutions of the optimal transportation problem on domains with boundary. *J. Reine Angew. Math.* **672** 127–160. MR2995434 https://doi.org/10.1515/crelle.2012.001

[46] KNIGHT, P. A. (2008). The Sinkhorn-Knopp algorithm: Convergence and applications. *SIAM J. Matrix Anal. Appl.* **30** 261–275. MR2399579 https://doi.org/10.1137/060659624

[47] KOLESNIKOV, A. V. (2014). Hessian metrics, $CD(K, N)$-spaces, and optimal transportation of log-concave measures. *Discrete Contin. Dyn. Syst.* **34** 1511–1532. MR3121630 https://doi.org/10.3934/dcds.2014.34.1511

[48] LAURENT, B. and MASSART, P. (2000). Adaptive estimation of a quadratic functional by model selection. *Ann. Statist.* **28** 1302–1338. MR1805785 https://doi.org/10.1214/aos/1015957395

[49] LÉGER, F. (2021). A gradient descent perspective on Sinkhorn. *Appl. Math. Optim.* **84** 1843–1855. MR4304891 https://doi.org/10.1007/s00245-020-09697-w

[50] LEMONS, D. S. and GYTHIEL, A. (1997). Paul Langevin's 1908 paper "On the theory of Brownian motion" ["Sur la théorie du mouvement Brownien," CR Acad. Sci. (Paris) 146, 530–533 (1908)]. *Amer. J. Phys.* **65** 1079–1081.

[51] LÉONARD, C. (2012). From the Schrödinger problem to the Monge-Kantorovich problem. *J. Funct. Anal.* **262** 1879–1920. MR2873864 https://doi.org/10.1016/j.jfa.2011.11.026

[52] LIU, Q. (2017). Stein variational gradient descent as gradient flow. In *Advances in Neural Information Processing Systems* (I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan and R. Garnett, eds.) **30**. Curran Associates, Red Hook.

[53] MCCANN, R. J. (1995). Existence and uniqueness of monotone measure-preserving maps. *Duke Math. J.* **80** 309–323. MR1369395 https://doi.org/10.1215/S0012-7094-95-08013-2

[54] MCCANN, R. J. (2001). Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.* **11** 589–608. MR1844080 https://doi.org/10.1007/PL00001679

[55] MCKEAN, H. P. (1975). Fluctuations in the kinetic theory of gases. *Comm. Pure Appl. Math.* **28** 435–455. MR0395662 https://doi.org/10.1002/cpa.3160280402

[56] MCKEAN, H. P. JR. (1966). A class of Markov processes associated with nonlinear parabolic equations. *Proc. Natl. Acad. Sci. USA* **56** 1907–1911. MR0221595 https://doi.org/10.1073/pnas.56.6.1907

[57] MIKAMI, T. (2004). Monge's problem with a quadratic cost by the zero-noise limit of h-path processes. *Probab. Theory Related Fields* **129** 245–260. MR2063377 https://doi.org/10.1007/s00440-004-0340-4

[58] MONGE, G. (1781). Mémoire sur la théorie des déblais et des remblais. *Mem. Math. Phys. Acad. R. Sci.* 666–704.

[59] MOOSMÜLLER, C. and CLONINGER, A. (2023). Linear optimal transport embedding: Provable Wasserstein classification for certain rigid transformations and perturbations. *Inf. Inference* **12** 363–389. MR4565740 https://doi.org/10.1093/imaiai/iaac023

[60] NEMIROVSKY, A. S. and YUDIN, D. B. (1983). *Problem Complexity and Method Efficiency in Optimization*. *Wiley-Interscience Series in Discrete Mathematics*. Wiley, New York. MR0702836

[61] NUTZ, M. and WIESEL, J. (2022). Entropic optimal transport: Convergence of potentials. *Probab. Theory Related Fields* **184** 401–424. MR4498514 https://doi.org/10.1007/s00440-021-01096-8
[62] NUTZ, M. and WIESEL, J. (2023). Stability of Schrödinger potentials and convergence of Sinkhorn's algorithm. *Ann. Probab.* **51** 699–722. MR4546630 https://doi.org/10.1214/22-aop1611
[63] OTTO, F. (2001). The geometry of dissipative evolution equations: The porous medium equation. *Comm. Partial Differential Equations* **26** 101–174. MR1842429 https://doi.org/10.1081/PDE-100002243
[64] OTTO, F. and VILLANI, C. (2000). Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.* **173** 361–400. MR1760620 https://doi.org/10.1006/jfan.1999.3557
[65] PAL, S. (2024). On the difference between entropic cost and the optimal transport cost. *Ann. Appl. Probab.* **34** 1003–1028. MR4700251 https://doi.org/10.1214/23-aap1983
[66] PEYRÉ, G., CUTURI, M. et al. (2019). Computational optimal transport: With applications to data science. *Found. Trends Mach. Learn.* **11** 355–607.
[67] RANKIN, C. and WONG, L. (2023). Bregman-Wasserstein divergence: Geometry and applications. Preprint. Available at arXiv:2302.05833.
[68] ROBERTS, G. O. and TWEEDIE, R. L. (1996). Exponential convergence of Langevin distributions and their discrete approximations. *Bernoulli* **2** 341–363. MR1440273 https://doi.org/10.2307/3318418
[69] RUBNER, Y., TOMASI, C. and GUIBAS, L. J. (2000). The earth mover's distance as a metric for image retrieval. *Int. J. Comput. Vis.* **40** 99.
[70] RÜSCHENDORF, L. (1995). Convergence of the iterative proportional fitting procedure. *Ann. Statist.* **23** 1160–1174. MR1353500 https://doi.org/10.1214/aos/1176324703
[71] SANTAMBROGIO, F. (2015). *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling. Progress in Nonlinear Differential Equations and Their Applications* **87**. Birkhäuser/Springer, Cham. MR3409718 https://doi.org/10.1007/978-3-319-20828-2
[72] SCHIEBINGER, G., SHU, J., TABAKA, M., CLEARY, B., SUBRAMANIAN, V., SOLOMON, A., GOULD, J., LIU, S., LIN, S. et al. (2019). Optimal-transport analysis of single-cell gene expression identifies developmental trajectories in reprogramming. *Cell* **176** 928–943.
[73] SCHRÖDINGER, E. (1932). Sur la théorie relativiste de l'électron et l'interprétation de la mécanique quantique. *Ann. Inst. Henri Poincaré* **2** 269–310. MR1508000
[74] SHARROCK, L., MACKEY, L. and NEMETH, C. (2023). Learning rate free Bayesian inference in constrained domains. Preprint. Available at arXiv:2305.14943.
[75] SHIMA, H. and YAGI, K. (1997). Geometry of Hessian manifolds. *Differential Geom. Appl.* **7** 277–290. MR1480540 https://doi.org/10.1016/S0926-2245(96)00057-5
[76] SINKHORN, R. (1967). Diagonal equivalence to matrices with prescribed row and column sums. *Amer. Math. Monthly* **74** 402–405. MR0210730 https://doi.org/10.2307/2314570
[77] SOHL-DICKSTEIN, J., WEISS, E., MAHESWARANATHAN, N. and GANGULI, S. (2015). Deep unsupervised learning using nonequilibrium thermodynamics. In *International Conference on Machine Learning* 2256–2265. PMLR.
[78] SONG, Y., SOHL-DICKSTEIN, J., KINGMA, D. P., KUMAR, A., ERMON, S. and POOLE, B. (2020). Score-based generative modeling through stochastic differential equations. Preprint. Available at arXiv:2011.13456.
[79] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX—1989. Lecture Notes in Math.* **1464** 165–251. Springer, Berlin. MR1108185 https://doi.org/10.1007/BFb0085169
[80] TANAKA, H. (1978/79). Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Gebiete* **46** 67–105. MR0512334 https://doi.org/10.1007/BF00535689
[81] TANG, L. (2013). Regularity results on the parabolic Monge–Ampère equation with *VMO* type data. *J. Differ. Equ.* **255** 1646–1656. MR3072666 https://doi.org/10.1016/j.jde.2013.05.019
[82] VILLANI, C. (2003). *Topics in Optimal Transportation. Graduate Studies in Mathematics* **58**. Amer. Math. Soc., Providence, RI. MR1964483 https://doi.org/10.1090/gsm/058
[83] VILLANI, C. (2012). Optimal transportation, dissipative PDE's and functional inequalities. Available at https://cedricvillani.org/sites/dev/files/old_images/2012/08/B04.MFranca.pdf.
[84] WALTER, W. (1970). *Differential and Integral Inequalities. Ergebnisse der Mathematik und Ihrer Grenzgebiete [Results in Mathematics and Related Areas]* **55**. Springer, New York. MR0271508
[85] WANG, F.-Y. (2001). Logarithmic Sobolev inequalities: Conditions and counterexamples. *J. Operator Theory* **46** 183–197. MR1862186
[86] WANG, W., SLEPČEV, D., BASU, S., OZOLEK, J. A. and ROHDE, G. K. (2013). A linear optimal transportation framework for quantifying and visualizing variations in sets of images. *Int. J. Comput. Vis.* **101** 254–269. MR3021062 https://doi.org/10.1007/s11263-012-0566-z

[87] WANG, Y. and LI, W. (2020). Information Newton's flow: Second-order optimization method in probability space. Preprint. Available at arXiv:2001.04341.
[88] WILSON, A. (2018). Lyapunov arguments in optimization. Ph.D. thesis, UC Berkeley. Available at https://escholarship.org/uc/item/1116c975.
[89] ZHANG, K. S., PEYRÉ, G., FADILI, J. and PEREYRA, M. (2020). Wasserstein control of mirror Langevin Monte Carlo. In *Conference on Learning Theory* 3814–3841. PMLR.

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ASYMPTOTIC EXPANSION OF SMOOTH FUNCTIONS IN DETERMINISTIC AND IID HAAR UNITARY MATRICES, AND APPLICATION TO TENSOR PRODUCTS OF MATRICES

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Let U^N be a family of $N \times N$ independent Haar unitary random matrices and their adjoints, Z^N a family of deterministic matrices, and P a self-adjoint noncommutative polynomial, that is, for any N, $P(U^N, Z^N)$ is self-adjoint, f a smooth function. We prove that for any k, if f is smooth enough, there exist deterministic constants $\alpha_i^P(f, Z^N)$ such that

$$\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\big(f\big(P(U^N, Z^N)\big)\big)\right] = \sum_{i=0}^{k} \frac{\alpha_i^P(f, Z^N)}{N^{2i}} + \mathcal{O}(N^{-2k-2}).$$

Besides, the constants $\alpha_i^P(f, Z^N)$ are built explicitly with the help of free probability. As a corollary, we prove that given $\alpha < 1/2$, for N large enough, every eigenvalue of $P(U^N, Z^N)$ is $N^{-\alpha}$-close to the spectrum of $P(u, Z^N)$ where u is a d-tuple of free Haar unitaries. We also prove the convergence of the norm of any polynomial $P(U^N \otimes I_M, I_N \otimes Y^M)$ as long as the family Y^M converges strongly and that $M \ll N \ln^{-3}(N)$.

REFERENCES

[1] ALBEVERIO, S., PASTUR, L. and SHCHERBINA, M. (2001). On the $1/n$ expansion for some unitary invariant ensembles of random matrices. *Comm. Math. Phys.* **224** 271–305.

[2] ANDERSON, G. W., GUIONNET, A. and ZEITOUNI, O. (2010). *An Introduction to Random Matrices*. *Cambridge Studies in Advanced Mathematics* **118**. Cambridge Univ. Press, Cambridge. MR2760897

[3] BANDEIRA, A. S., BOEDIHARDJO, M. T. and VAN HANDEL, R. (2023). Matrix concentration inequalities and free probability. *Invent. Math.* **234** 419–487. MR4635836 https://doi.org/10.1007/s00222-023-01204-6

[4] BELINSCHI, S. and CAPITAINE, M. (2022). Strong convergence of tensor products of independent GUE matrices. arXiv preprint. Available at arXiv:2205.07695.

[5] BIANE, P. (1997). Free Brownian motion, free stochastic calculus and random matrices. In *Free Probability Theory* (*Waterloo*, *ON*, 1995). *Fields Inst. Commun.* **12** 1–19. Amer. Math. Soc., Providence, RI. MR1426833 https://doi.org/10.1090/fic/012/01

[6] BIANE, P. (1997). Segal–Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems. *J. Funct. Anal.* **144** 232–286. MR1430721 https://doi.org/10.1006/jfan.1996.2990

[7] BORDENAVE, C. and COLLINS, B. (2023). Norm of matrix-valued polynomials in random unitaries and permutations. arXiv preprint. Available at arXiv:2304.05714.

[8] BOROT, G. and GUIONNET, A. (2013). Asymptotic expansion of β matrix models in the one-cut regime. *Comm. Math. Phys.* **317** 447–483. MR3010191 https://doi.org/10.1007/s00220-012-1619-4

[9] BOROT, G. and GUIONNET, A. (2024). Asymptotic expansion of β matrix models in the multi-cut regime. *Forum Math. Sigma* **12** Paper No. e13. MR4695502 https://doi.org/10.1017/fms.2023.129

[10] BOROT, G., GUIONNET, A. and KOZLOWSKI, K. K. (2015). Large-N asymptotic expansion for mean field models with Coulomb gas interaction. *Int. Math. Res. Not. IMRN* **20** 10451–10524. MR3455872 https://doi.org/10.1093/imrn/rnu260

[11] BOROT, G., GUIONNET, A. and KOZLOWSKI, K. K. (2016). *Asymptotic Expansion of a Partition Function Related to the Sinh-Model*. *Mathematical Physics Studies*. Springer, Cham. MR3585351 https://doi.org/10.1007/978-3-319-33379-3

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[12] BRÉZIN, E., ITZYKSON, C., PARISI, G. and ZUBER, J. B. (1993). Planar diagrams. In *The Large N Expansion in Quantum Field Theory and Statistical Physics*: *From Spin Systems to* 2*-Dimensional Gravity* 567–583.

[13] BROWN, N. P. and OZAWA, N. (2008). *C^*-Algebras and Finite-Dimensional Approximations*. *Graduate Studies in Mathematics* **88**. Amer. Math. Soc., Providence, RI. MR2391387 https://doi.org/10.1090/gsm/088

[14] CHEKHOV, L. O., EYNARD, B. and MARCHAL, O. (2011). Topological expansion of the β-ensemble model and quantum algebraic geometry in the sectorwise approach. *Theoret. and Math. Phys.* **166** 141–185. MR3165804 https://doi.org/10.1007/s11232-011-0012-3

[15] CHEN, C.-F., GARZA-VARGAS, J., TROPP, J. A. and VAN HANDEL, R. (2026). A new approach to strong convergence. *Ann. of Math.* (2) **203**. MR5037839 https://doi.org/10.4007/annals.2026.203.2.4

[16] CHEN, C.-F., GARZA-VARGAS, J. and VAN HANDEL, R. (2024). A new approach to strong convergence II. The classical ensembles. arXiv preprint. Available at arXiv:2412.00593.

[17] CIPOLLONI, G., ERDŐS, L. and SCHRÖDER, D. (2022). Thermalisation for Wigner matrices. *J. Funct. Anal.* **282** Paper No. 109394. MR4372147 https://doi.org/10.1016/j.jfa.2022.109394

[18] COLLINS, B. (2003). Moments and cumulants of polynomial random variables on unitary groups, the Itzykson–Zuber integral, and free probability. *Int. Math. Res. Not.* **17** 953–982. MR1959915 https://doi.org/10.1155/S107379280320917X

[19] COLLINS, B., DAHLQVIST, A. and KEMP, T. (2018). The spectral edge of unitary Brownian motion. *Probab. Theory Related Fields* **170** 49–93. MR3748321 https://doi.org/10.1007/s00440-016-0753-x

[20] COLLINS, B., GUIONNET, A. and MAUREL-SEGALA, E. (2009). Asymptotics of unitary and orthogonal matrix integrals. *Adv. Math.* **222** 172–215. MR2531371 https://doi.org/10.1016/j.aim.2009.03.019

[21] COLLINS, B., GUIONNET, A. and PARRAUD, F. (2022). On the operator norm of non-commutative polynomials in deterministic matrices and iid GUE matrices. *Camb. J. Math.* **10** 195–260. MR4445344 https://doi.org/10.4310/cjm.2022.v10.n1.a3

[22] COLLINS, B. and MALE, C. (2014). The strong asymptotic freeness of Haar and deterministic matrices. *Ann. Sci. Éc. Norm. Supér.* (4) **47** 147–163. MR3205602 https://doi.org/10.24033/asens.2211

[23] COLLINS, B. and ŚNIADY, P. (2006). Integration with respect to the Haar measure on unitary, orthogonal and symplectic group. *Comm. Math. Phys.* **264** 773–795. MR2217291 https://doi.org/10.1007/s00220-006-1554-3

[24] DA SILVA, R. C. (2018). Lecture Notes on Noncommutative L_p-spaces. arXiv preprint. Available at arXiv:1803.02390.

[25] DRIVER, B. K., HALL, B. C. and KEMP, T. (2013). The large-N limit of the Segal–Bargmann transform on $\mathbb{U}_N$. *J. Funct. Anal.* **265** 2585–2644. MR3096985 https://doi.org/10.1016/j.jfa.2013.07.020

[26] ERCOLANI, N. M. and MCLAUGHLIN, K. D. T.-R. (2003). Asymptotics of the partition function for random matrices via Riemann–Hilbert techniques and applications to graphical enumeration. *Int. Math. Res. Not.* **14** 755–820. MR1953782 https://doi.org/10.1155/S1073792803211089

[27] FRONK, J., KRÜGER, T. and NEMISH, Y. (2024). Norm convergence rate for multivariate quadratic polynomials of Wigner matrices. *J. Funct. Anal.* **287** Paper No. 110647. MR4796149 https://doi.org/10.1016/j.jfa.2024.110647

[28] GUIONNET, A. (2019). *Asymptotics of Random Matrices and Related Models*: *The Uses of Dyson–Schwinger Equations*. *CBMS Regional Conference Series in Mathematics* **130**. Amer. Math. Soc., Providence, RI. MR3931304 https://doi.org/10.1090/cbms/130

[29] GUIONNET, A. and MAUREL-SEGALA, E. (2006). Combinatorial aspects of matrix models. *ALEA Lat. Amer. J. Probab. Math. Stat.* **1** 241–279. MR2249657

[30] GUIONNET, A. and MAUREL-SEGALA, E. (2007). Second order asymptotics for matrix models. *Ann. Probab.* **35** 2160–2212. MR2353386 https://doi.org/10.1214/009117907000000141

[31] GUIONNET, A. and NOVAK, J. (2015). Asymptotics of unitary multimatrix models: The Schwinger–Dyson lattice and topological recursion. *J. Funct. Anal.* **268** 2851–2905. MR3331788 https://doi.org/10.1016/j.jfa.2015.03.002

[32] HAAGERUP, U. and THORBJØRNSEN, S. (2005). A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(F_2))$ is not a group. *Ann. of Math.* (2) **162** 711–775. MR2183281 https://doi.org/10.4007/annals.2005.162.711

[33] HAAGERUP, U. and THORBJØRNSEN, S. (2012). Asymptotic expansions for the Gaussian unitary ensemble. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **15** 1250003. MR2922846 https://doi.org/10.1142/S0219025712500038

[34] HARER, J. and ZAGIER, D. (1986). The Euler characteristic of the moduli space of curves. *Invent. Math.* **85** 457–485. MR0848681 https://doi.org/10.1007/BF01390325

[35] HAYES, B. (2022). A random matrix approach to the Peterson–Thom conjecture. *Indiana Univ. Math. J.* **71** 1243–1297. MR4448584

[36] LE GALL, J.-F. (2016). *Brownian Motion, Martingales, and Stochastic Calculus*, French ed. *Graduate Texts in Mathematics* **274**. Springer, Cham. MR3497465 https://doi.org/10.1007/978-3-319-31089-3

[37] MAGEE, M. and DE LA SALLE, M. (2026). Strong Asymptotic Freeness of Haar Unitaries in Quasi-Exponential Dimensional Representations. *Geom. Funct. Anal.* **36** 152–200. MR5033337 https://doi.org/10.1007/s00039-026-00730-8

[38] MALE, C. (2012). The norm of polynomials in large random and deterministic matrices. *Probab. Theory Related Fields* **154** 477–532. MR3000553 https://doi.org/10.1007/s00440-011-0375-2

[39] MATSUMOTO, S. and NOVAK, J. (2013). Jucys–Murphy elements and unitary matrix integrals. *Int. Math. Res. Not. IMRN* **2** 362–397. MR3010693 https://doi.org/10.1093/imrn/rnr267

[40] MECKES, E. S. (2019). *The Random Matrix Theory of the Classical Compact Groups*. *Cambridge Tracts in Mathematics* **218**. Cambridge Univ. Press, Cambridge. MR3971582 https://doi.org/10.1017/9781108303453.009

[41] MURPHY, G. J. (1990). *C^*-Algebras and Operator Theory*. Academic Press, Boston, MA. MR1074574

[42] NICA, A. and SPEICHER, R. (2006). *Lectures on the Combinatorics of Free Probability*. *London Mathematical Society Lecture Note Series* **335**. Cambridge Univ. Press, Cambridge. MR2266879 https://doi.org/10.1017/CBO9780511735127

[43] NOVAK, J. (2020). On the complex asymptotics of the HCIZ and BGW integrals. arXiv preprint. Available at arXiv:2006.04304.

[44] PARRAUD, F. (2022). On the operator norm of non-commutative polynomials in deterministic matrices and iid Haar unitary matrices. *Probab. Theory Related Fields* **182** 751–806. MR4408503 https://doi.org/10.1007/s00440-021-01101-0

[45] PARRAUD, F. (2023). Asymptotic expansion of smooth functions in polynomials in deterministic matrices and iid GUE matrices. *Comm. Math. Phys.* **399** 249–294. MR4567374 https://doi.org/10.1007/s00220-022-04551-2

[46] PARRAUD, F. (2024). The spectrum of a tensor of random and deterministic matrices. arXiv preprint. Available at arXiv:2410.04481.

[47] PARRAUD, F. and SCHNELLI, K. (2024). Asymptotic freeness through unitaries generated by polynomials of Wigner matrices. *Linear Algebra Appl.* **699** 1–46. MR4767467 https://doi.org/10.1016/j.laa.2024.06.014

[48] PÉCHÉ, S. (2006). The largest eigenvalue of small rank perturbations of Hermitian random matrices. *Probab. Theory Related Fields* **134** 127–173. MR2221787 https://doi.org/10.1007/s00440-005-0466-z

[49] PISIER, G. (2003). *Introduction to Operator Space Theory*. *London Mathematical Society Lecture Note Series* **294**. Cambridge Univ. Press, Cambridge. MR2006539 https://doi.org/10.1017/CBO9781107360235

[50] PISIER, G. (2014). Random matrices and subexponential operator spaces. *Israel J. Math.* **203** 223–273. MR3273440 https://doi.org/10.1007/s11856-014-1069-0

[51] SEGALA, E. M. (2006). High order asymptotics of matrix models and enumeration of maps. arXiv preprint. Available at arXiv:math/0608192.

[52] SHCHERBINA, M. (2014). Asymptotic expansions for *beta* matrix models and their applications to the universality conjecture. In *Random Matrix Theory, Interacting Particle Systems, and Integrable Systems*. *Math. Sci. Res. Inst. Publ.* **65** 463–482. Cambridge Univ. Press, New York. MR3380697

[53] T'HOOFT, G. (1974). Magnetic monopoles in unified gauge theories. *Nucl. Phys.* **B79** 276–284. MR0413809 https://doi.org/10.1016/0550-3213(74)90486-6

[54] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246

[55] VOICULESCU, D. (1991). Limit laws for random matrices and free products. *Invent. Math.* **104** 201–220. MR1094052 https://doi.org/10.1007/BF01245072

[56] VOICULESCU, D. (1993). The analogues of entropy and of Fisher's information measure in free probability theory. I. *Comm. Math. Phys.* **155** 71–92. MR1228526

[57] VOICULESCU, D. (1998). A strengthened asymptotic freeness result for random matrices with applications to free entropy. *Int. Math. Res. Not.* **1** 41–63. MR1601878 https://doi.org/10.1155/S107379289800004X

[58] VOICULESCU, D. (1999). The analogues of entropy and of Fisher's information measure in free probability theory. VI. Liberation and mutual free information. *Adv. Math.* **146** 101–166. MR1711843 https://doi.org/10.1006/aima.1998.1819

[59] VOICULESCU, D. (2000). The coalgebra of the free difference quotient and free probability. *Int. Math. Res. Not.* **2** 79–106. MR1744647 https://doi.org/10.1155/S1073792800000064

[60] WEINGARTEN, D. (1978). Asymptotic behavior of group integrals in the limit of infinite rank. *J. Math. Phys.* **19** 999–1001. MR0471696 https://doi.org/10.1063/1.523807

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EDGE UNIVERSALITY OF RANDOM REGULAR GRAPHS OF GROWING DEGREES

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We consider the statistics of extreme eigenvalues of random d-regular graphs, with $N^{\mathfrak{c}} \le d \le N^{1/3-\mathfrak{c}}$ for arbitrarily small $\mathfrak{c} > 0$. We prove that in this regime, the fluctuations of extreme eigenvalues are given by the Tracy–Widom distribution. As a consequence, about 69% of d-regular graphs have all nontrivial eigenvalues bounded in absolute value by $2\sqrt{d-1}$.

REFERENCES

[1] ADHIKARI, A. and HUANG, J. (2020). Dyson Brownian motion for general β and potential at the edge. *Probab. Theory Related Fields* **178** 893–950. MR4168391 https://doi.org/10.1007/s00440-020-00992-9
[2] AIZENMAN, M. and WARZEL, S. (2015). *Random Operators*. *Graduate Studies in Mathematics* **168**. Amer. Math. Soc., Providence, RI. Disorder effects on quantum spectra and dynamics. MR3364516 https://doi.org/10.1090/gsm/168
[3] ALON, N. (1986). Eigenvalues and expanders. *Combinatorica* **6** 83–96. Theory of computing (Singer Island, Fla., 1984). https://doi.org/10.1007/BF02579166
[4] ALT, J., DUCATEZ, R. and KNOWLES, A. (2021). Extremal eigenvalues of critical Erdős–Rényi graphs. *Ann. Probab.* **49** 1347–1401. MR4255147 https://doi.org/10.1214/20-aop1483
[5] ALT, J., DUCATEZ, R. and KNOWLES, A. (2021). Delocalization transition for critical Erdős–Rényi graphs. *Comm. Math. Phys.* **388** 507–579. MR4328063 https://doi.org/10.1007/s00220-021-04167-y
[6] ALT, J., DUCATEZ, R. and KNOWLES, A. (2022). The completely delocalized region of the Erdős–Rényi graph. *Electron. Commun. Probab.* **27** 10. MR4375917 https://doi.org/10.1214/22-ECP450
[7] ALT, J., DUCATEZ, R. and KNOWLES, A. (2023). Poisson statistics and localization at the spectral edge of sparse Erdős–Rényi graphs. *Ann. Probab.* **51** 277–358. MR4515695 https://doi.org/10.1214/22-aop1596
[8] BAUERSCHMIDT, R., HUANG, J., KNOWLES, A. and YAU, H.-T. (2017). Bulk eigenvalue statistics for random regular graphs. *Ann. Probab.* **45** 3626–3663. MR3729611 https://doi.org/10.1214/16-AOP1145
[9] BAUERSCHMIDT, R., HUANG, J., KNOWLES, A. and YAU, H.-T. (2020). Edge rigidity and universality of random regular graphs of intermediate degree. *Geom. Funct. Anal.* **30** 693–769. MR4135670 https://doi.org/10.1007/s00039-020-00538-0
[10] BAUERSCHMIDT, R., HUANG, J. and YAU, H.-T. (2019). Local Kesten–McKay law for random regular graphs. *Comm. Math. Phys.* **369** 523–636. MR3962004 https://doi.org/10.1007/s00220-019-03345-3
[11] BAUERSCHMIDT, R., KNOWLES, A. and YAU, H.-T. (2017). Local semicircle law for random regular graphs. *Comm. Pure Appl. Math.* **70** 1898–1960. MR3688032 https://doi.org/10.1002/cpa.21709
[12] BENAYCH-GEORGES, F., BORDENAVE, C. and KNOWLES, A. (2019). Largest eigenvalues of sparse inhomogeneous Erdős–Rényi graphs. *Ann. Probab.* **47** 1653–1676. MR3945756 https://doi.org/10.1214/18-AOP1293
[13] BENAYCH-GEORGES, F., BORDENAVE, C. and KNOWLES, A. (2020). Spectral radii of sparse random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **56** 2141–2161. MR4116720 https://doi.org/10.1214/19-AIHP1033
[14] BIANE, P. (1997). On the free convolution with a semi-circular distribution. *Indiana Univ. Math. J.* **46** 705–718. MR1488333 https://doi.org/10.1512/iumj.1997.46.1467
[15] BORDENAVE, C. (2020). A new proof of Friedman's second eigenvalue theorem and its extension to random lifts. *Ann. Sci. Éc. Norm. Supér.* (4) **53** 1393–1439. MR4203039 https://doi.org/10.24033/asens.2450

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[16] BOURGADE, P., ERDŐS, L., YAU, H.-T. and YIN, J. (2016). Fixed energy universality for generalized Wigner matrices. *Comm. Pure Appl. Math.* **69** 1815–1881. MR3541852 https://doi.org/10.1002/cpa.21624

[17] BOURGADE, P., HUANG, J. and YAU, H.-T. (2017). Eigenvector statistics of sparse random matrices. *Electron. J. Probab.* **22** 64. MR3690289 https://doi.org/10.1214/17-EJP81

[18] BOURGADE, P. and YAU, H.-T. (2017). The eigenvector moment flow and local quantum unique ergodicity. *Comm. Math. Phys.* **350** 231–278. MR3606475 https://doi.org/10.1007/s00220-016-2627-6

[19] CARR, C. (2025). CDF for Tracy–Widom (TW1) distribution. https://www.mathworks.com/matlabcentral/fileexchange/30983-cdf-for-tracy-widom-tw1-distribution. MATLAB Central File Exchange. Retrieved 2025-09-09.

[20] CHEN, C.-F., GARZA-VARGAS, J., TROPP, J. A. and VAN HANDEL, R. (2026). A new approach to strong convergence. *Ann. of Math.* (2) **203**. MR5037839 https://doi.org/10.4007/annals.2026.203.2.4

[21] CHEN, C.-F., GARZA-VARGAS, J. and VAN HANDEL, R. (2024). A new approach to strong convergence II. The classical ensembles. arXiv preprint. Available at arXiv:2412.00593.

[22] COHEN, M. B. (2016). Ramanujan graphs in polynomial time. In 57*th Annual IEEE Symposium on Foundations of Computer Science—FOCS* 2016 276–281. IEEE Computer Soc., Los Alamitos, CA. MR3630988 https://doi.org/10.1109/FOCS.2016.37

[23] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2012). Spectral statistics of Erdős–Rényi Graphs II: Eigenvalue spacing and the extreme eigenvalues. *Comm. Math. Phys.* **314** 587–640. MR2964770 https://doi.org/10.1007/s00220-012-1527-7

[24] ERDŐS, L., KNOWLES, A., YAU, H.-T. and YIN, J. (2013). Spectral statistics of Erdős–Rényi graphs I: Local semicircle law. *Ann. Probab.* **41** 2279–2375. MR3098073 https://doi.org/10.1214/11-AOP734

[25] ERDŐS, L., SCHLEIN, B. and YAU, H.-T. (2011). Universality of random matrices and local relaxation flow. *Invent. Math.* **185** 75–119. MR2810797 https://doi.org/10.1007/s00222-010-0302-7

[26] ERDŐS, L., SCHLEIN, B., YAU, H.-T. and YIN, J. (2012). The local relaxation flow approach to universality of the local statistics for random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 1–46. MR2919197 https://doi.org/10.1214/10-AIHP388

[27] ERDŐS, L. and SCHNELLI, K. (2017). Universality for random matrix flows with time-dependent density. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 1606–1656. MR3729630 https://doi.org/10.1214/16-AIHP765

[28] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory*. *Courant Lecture Notes in Mathematics* **28**. Courant Institute of Mathematical Sciences, New York; Amer. Math. Soc., Providence, RI. MR3699468

[29] ERDŐS, L. and YAU, H.-T. (2017). *A Dynamical Approach to Random Matrix Theory*. *Courant Lecture Notes in Mathematics* **28**. Courant Institute of Mathematical Sciences, New York; Amer. Math. Soc., Providence, RI. MR3699468

[30] ERDŐS, L., YAU, H.-T. and YIN, J. (2012). Rigidity of eigenvalues of generalized Wigner matrices. *Adv. Math.* **229** 1435–1515. MR2871147 https://doi.org/10.1016/j.aim.2011.12.010

[31] FELDHEIM, O. N. and SODIN, S. (2010). A universality result for the smallest eigenvalues of certain sample covariance matrices. *Geom. Funct. Anal.* **20** 88–123. MR2647136 https://doi.org/10.1007/s00039-010-0055-x

[32] FRIEDMAN, J. (2003). A proof of Alon's second eigenvalue conjecture. In *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing* 720–724. ACM, New York. MR2120428 https://doi.org/10.1145/780542.780646

[33] HE, Y. (2024). Spectral gap and edge universality of dense random regular graphs. *Comm. Math. Phys.* **405** 181. MR4777082 https://doi.org/10.1007/s00220-024-05063-x

[34] HE, Y. and KNOWLES, A. (2017). Mesoscopic eigenvalue statistics of Wigner matrices. *Ann. Appl. Probab.* **27** 1510–1550. MR3678478 https://doi.org/10.1214/16-AAP1237

[35] HE, Y. and KNOWLES, A. (2021). Fluctuations of extreme eigenvalues of sparse Erdős–Rényi graphs. *Probab. Theory Related Fields* **180** 985–1056. MR4288336 https://doi.org/10.1007/s00440-021-01054-4

[36] HOORY, S., LINIAL, N. and WIGDERSON, A. (2006). Expander graphs and their applications. *Bull. Amer. Math. Soc.* (*N.S.*) **43** 439–561. MR2247919 https://doi.org/10.1090/S0273-0979-06-01126-8

[37] HUANG, J., LANDON, B. and YAU, H.-T. (2015). Bulk universality of sparse random matrices. *J. Math. Phys.* **56** 123301. MR3429490 https://doi.org/10.1063/1.4936139

[38] HUANG, J., LANDON, B. and YAU, H.-T. (2020). Transition from Tracy-Widom to Gaussian fluctuations of extremal eigenvalues of sparse Erdős–Rényi graphs. *Ann. Probab.* **48** 916–962. MR4089498 https://doi.org/10.1214/19-AOP1378

[39] HUANG, J., MCKENZIE, T. and YAU, H.-T. (2024). Ramanujan property and edge universality of random regular graphs. arXiv preprint. Available at arXiv:2412.20263.

[40] HUANG, J. and YAU, H.-T. (2024). Spectrum of random d-regular graphs up to the edge. *Comm. Pure Appl. Math.* **77** 1635–1723. MR4692880 https://doi.org/10.1002/cpa.22176

[41] HUANG, J. and YAU, H.-T. (2026). Edge universality of sparse random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **62**. MR5036525 https://doi.org/10.1214/24-aihp1474

[42] KESTEN, H. (1959). Symmetric random walks on groups. *Trans. Amer. Math. Soc.* **92** 336–354. MR0109367 https://doi.org/10.2307/1993160

[43] KHORUNZHIY, O. (2012). High moments of large Wigner random matrices and asymptotic properties of the spectral norm. *Random Oper. Stoch. Equ.* **20** 25–68. MR2899796 https://doi.org/10.1515/rose-2012-0002

[44] KNOWLES, A. and YIN, J. (2013). Eigenvector distribution of Wigner matrices. *Probab. Theory Related Fields* **155** 543–582. MR3034787 https://doi.org/10.1007/s00440-011-0407-y

[45] LANDON, B., SOSOE, P. and YAU, H.-T. (2019). Fixed energy universality of Dyson Brownian motion. *Adv. Math.* **346** 1137–1332. MR3914908 https://doi.org/10.1016/j.aim.2019.02.010

[46] LANDON, B. and YAU, H.-T. (2017). Edge statistics of Dyson Brownian motion. *Electron. J. Probab.* To appear.

[47] LANDON, B. and YAU, H.-T. (2017). Convergence of local statistics of Dyson Brownian motion. *Comm. Math. Phys.* **355** 949–1000. MR3687212 https://doi.org/10.1007/s00220-017-2955-1

[48] LEE, J. (2024). Higher order fluctuations of extremal eigenvalues of sparse random matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **60** 2694–2735. MR4828855 https://doi.org/10.1214/23-aihp1398

[49] LEE, J. O. and SCHNELLI, K. (2018). Local law and Tracy-Widom limit for sparse random matrices. *Probab. Theory Related Fields* **171** 543–616. MR3800840 https://doi.org/10.1007/s00440-017-0787-8

[50] LEE, J. O. and YIN, J. (2014). A necessary and sufficient condition for edge universality of Wigner matrices. *Duke Math. J.* **163** 117–173. MR3161313 https://doi.org/10.1215/00127094-2414767

[51] LUBOTZKY, A., PHILLIPS, R. and SARNAK, P. (1988). Ramanujan graphs. *Combinatorica* **8** 261–277. MR0963118 https://doi.org/10.1007/BF02126799

[52] MARCUS, A., SPIELMAN, D. A. and SRIVASTAVA, N. (2013). Interlacing families I: Bipartite Ramanujan graphs of all degrees. In 2013 *IEEE* 54*th Annual Symposium on Foundations of Computer Science—FOCS* 2013 529–537. IEEE Computer Soc., Los Alamitos, CA. MR3246256 https://doi.org/10.1109/FOCS.2013.63

[53] MARCUS, A. W., SPIELMAN, D. A. and SRIVASTAVA, N. (2018). Interlacing families IV: Bipartite Ramanujan graphs of all sizes. *SIAM J. Comput.* **47** 2488–2509. MR3892446 https://doi.org/10.1137/16M106176X

[54] MARGULIS, G. A. (1988). Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii* **24** 51–60. MR0939574

[55] MCKAY, B. D., WORMALD, N. C. and WYSOCKA, B. (2004). Short cycles in random regular graphs. *Electron. J. Combin.* **11** 66. MR2097332 https://doi.org/10.37236/1819

[56] MILLER, S. J. and NOVIKOFF, T. (2008). The distribution of the largest nontrivial eigenvalues in families of random regular graphs. *Exp. Math.* **17** 231–244. MR2433888

[57] PÉCHÉ, S. (2009). Universality results for the largest eigenvalues of some sample covariance matrix ensembles. *Probab. Theory Related Fields* **143** 481–516. MR2475670 https://doi.org/10.1007/s00440-007-0133-7

[58] RUZMAIKINA, A. (2006). Universality of the edge distribution of eigenvalues of Wigner random matrices with polynomially decaying distributions of entries. *Comm. Math. Phys.* **261** 277–296. MR2191882 https://doi.org/10.1007/s00220-005-1386-6

[59] SARNAK, P. (2004). What is... an expander? *Notices Amer. Math. Soc.* **51** 762–763. MR2072849

[60] SODIN, S. (2010). The spectral edge of some random band matrices. *Ann. of Math.* (2) **172** 2223–2251. MR2726110 https://doi.org/10.4007/annals.2010.172.2223

[61] SOSHNIKOV, A. (1999). Universality at the edge of the spectrum in Wigner random matrices. *Comm. Math. Phys.* **207** 697–733. MR1727234 https://doi.org/10.1007/s002200050743

[62] TAO, T. and VU, V. (2010). Random matrices: Universality of local eigenvalue statistics up to the edge. *Comm. Math. Phys.* **298** 549–572. MR2669449 https://doi.org/10.1007/s00220-010-1044-5

[63] TIKHOMIROV, K. and YOUSSEF, P. (2021). Outliers in spectrum of sparse Wigner matrices. *Random Structures Algorithms* **58** 517–605. MR4234995 https://doi.org/10.1002/rsa.20982

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TIGHTNESS OF THE MAXIMUM OF GINZBURG–LANDAU FIELDS

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We consider the discrete Ginzburg–Landau field with potential satisfying a uniform convexity condition, in the critical dimension $d = 2$, and prove that its maximum over boxes of sidelength N, centered by an explicit N-dependent centering, is tight.

REFERENCES

[1] ADDARIO-BERRY, L. and REED, B. A. (2008). Ballot theorems, old and new. In *Horizons of Combinatorics*. *Bolyai Soc. Math. Stud.* **17** 9–35. Springer, Berlin. MR2432525 https://doi.org/10.1007/978-3-540-77200-2_1

[2] ARGUIN, L.-P., BELIUS, D. and BOURGADE, P. (2017). Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.* **349** 703–751. MR3594368 https://doi.org/10.1007/s00220-016-2740-6

[3] ARGUIN, L.-P., BELIUS, D., BOURGADE, P., RADZIWIŁŁ, M. and SOUNDARARAJAN, K. (2019). Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure Appl. Math.* **72** 500–535. MR3911893 https://doi.org/10.1002/cpa.21791

[4] ARGUIN, L. P., BOURGADE, P. and RADZIWIŁŁ, M. (2020). The Fyodorov-Hiary-Keating Conjecture. I, II. Available at arXiv:2007.00988, arXiv:2307.00982.

[5] ARMSTRONG, S. and DARIO, P. (2024). Quantitative hydrodynamic limits of the Langevin dynamics for gradient interface models. *Electron. J. Probab.* **29** Paper No. 9, 93. MR4689353 https://doi.org/10.1214/23-ejp1072

[6] ARMSTRONG, S. and WU, W. (2022). C^2 regularity of the surface tension for the $\nabla\phi$ interface model. *Comm. Pure Appl. Math.* **75** 349–421. MR4373172 https://doi.org/10.1002/cpa.22031

[7] BELIUS, D., ROSEN, J. and ZEITOUNI, O. (2020). Tightness for the cover time of the two dimensional sphere. *Probab. Theory Related Fields* **176** 1357–1437. MR4087494 https://doi.org/10.1007/s00440-019-00940-2

[8] BELIUS, D. and WU, W. (2020). Maximum of the Ginzburg–Landau fields. *Ann. Probab.* **48** 2647–2679. MR4164451 https://doi.org/10.1214/19-AOP1416

[9] BISKUP, M. (2020). Extrema of the two-dimensional discrete Gaussian free field. In *Random Graphs, Phase Transitions, and the Gaussian Free Field. Springer Proc. Math. Stat.* **304** 163–407. Springer, Cham. MR4043225 https://doi.org/10.1007/978-3-030-32011-9_3

[10] BISKUP, M. and LOUIDOR, O. (2018). Full extremal process, cluster law and freezing for the two-dimensional discrete Gaussian free field. *Adv. Math.* **330** 589–687. MR3787554 https://doi.org/10.1016/j.aim.2018.02.018

[11] BOBKOV, S. G. and CHISTYAKOV, G. P. (2015). On concentration functions of random variables. *J. Theoret. Probab.* **28** 976–988. MR3413964 https://doi.org/10.1007/s10959-013-0504-1

[12] BOLTHAUSEN, E., DEUSCHEL, J.-D. and GIACOMIN, G. (2001). Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.* **29** 1670–1692. MR1880237 https://doi.org/10.1214/aop/1015345767

[13] BRAMSON, M. (1983). Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.* **44** iv+190. MR0705746 https://doi.org/10.1090/memo/0285

[14] BRAMSON, M., DING, J. and ZEITOUNI, O. (2016). Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **69** 62–123. MR3433630 https://doi.org/10.1002/cpa.21621

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[15] Bramson, M., Ding, J. and Zeitouni, O. (2016). Convergence in law of the maximum of nonlattice branching random walk. *Ann. Inst. Henri Poincaré Probab. Stat.* **52** 1897–1924. MR3573300 https://doi.org/10.1214/15-AIHP703

[16] Brascamp, H. J. and Lieb, E. H. (1976). On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Funct. Anal.* **22** 366–389. MR0450480 https://doi.org/10.1016/0022-1236(76)90004-5

[17] Caddeo, P., Kim, Y. H. and Lubetzky, E. (2024). On level line fluctuations of SOS surfaces above a wall. *Forum Math. Sigma* **12** Paper No. e91, 59. MR4818942 https://doi.org/10.1017/fms.2024.91

[18] Caputo, P., Lubetzky, E., Martinelli, F., Sly, A. and Toninelli, F. L. (2016). Scaling limit and cube-root fluctuations in SOS surfaces above a wall. *J. Eur. Math. Soc.* (*JEMS*) **18** 931–995. MR3500829 https://doi.org/10.4171/JEMS/606

[19] Chhaibi, R., Madaule, T. and Najnudel, J. (2018). On the maximum of the CβE field. *Duke Math. J.* **167** 2243–2345. MR3848391 https://doi.org/10.1215/00127094-2018-0016

[20] Claeys, T., Fahs, B., Lambert, G. and Webb, C. (2021). How much can the eigenvalues of a random Hermitian matrix fluctuate? *Duke Math. J.* **170** 2085–2235. MR4278668 https://doi.org/10.1215/00127094-2020-0070

[21] Cortines, A., Hartung, L. and Louidor, O. (2019). Decorated random walk restricted to stay below a curve (supplement material). Available at arXiv:1902.10079.

[22] Cortines, A., Louidor, O. and Saglietti, S. (2021). A scaling limit for the cover time of the binary tree. *Adv. Math.* **391** Paper No. 107974, 78. MR4303734 https://doi.org/10.1016/j.aim.2021.107974

[23] Cosco, C., Nakajima, S. and Zeitouni, O. (2025). The maximum of the two dimensional Gaussian directed polymer in the subcritical regime. Available at arXiv:2503.17236.

[24] Dario, P. (2019). Quantitative homogenization of the disordered $\nabla\phi$ model. *Electron. J. Probab.* **24** Paper No. 90, 99. MR4003143 https://doi.org/10.1214/19-ejp347

[25] Dario, P. (2021–2022). Quantitative hydrodynamic limits of the Langevin dynamics for gradient interface models. In *Séminaire Laurent Schwartz—Équations aux Dérivées Partielles et Applications. Année* 2021–2022 Exp. No. **V**, 15. Inst. Hautes Études Sci., Bures-sur-Yvette. MR4632296

[26] Daviaud, O. (2006). Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.* **34** 962–986. MR2243875 https://doi.org/10.1214/009117906000000061

[27] Dellacherie, C. and Meyer, P.-A. (1982). *Probabilities and Potential B*: *Theory of Martingales*. *North-Holland Mathematics Studies* **72**. North-Holland, Amsterdam. Translated from the French by J. P. Wilson. MR0745449

[28] Dembo, A., Peres, Y., Rosen, J. and Zeitouni, O. (2004). Cover times for Brownian motion and random walks in two dimensions. *Ann. of Math.* (2) **160** 433–464. MR2123929 https://doi.org/10.4007/annals.2004.160.433

[29] Deuschel, J.-D. and Giacomin, G. (2000). Entropic repulsion for massless fields. *Stoch. Process. Appl.* **89** 333–354. MR1780295 https://doi.org/10.1016/S0304-4149(00)00030-2

[30] Deuschel, J.-D., Giacomin, G. and Ioffe, D. (2000). Large deviations and concentration properties for $\nabla\phi$ interface models. *Probab. Theory Related Fields* **117** 49–111. MR1759509 https://doi.org/10.1007/s004400050266

[31] Ding, J., Roy, R. and Zeitouni, O. (2017). Convergence of the centered maximum of log-correlated Gaussian fields. *Ann. Probab.* **45** 3886–3928. MR3729618 https://doi.org/10.1214/16-AOP1152

[32] Dolgopyat, D. (2016). A local limit theorem for sums of independent random vectors. *Electron. J. Probab.* **21** Paper No. 39, 15. MR3515569 https://doi.org/10.1214/16-EJP4232

[33] Fröhlich, J. and Spencer, T. (1981). The Kosterlitz–Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas. *Comm. Math. Phys.* **81** 527–602. MR0634447

[34] Funaki, T. (2005). Stochastic interface models. In *Lectures on Probability Theory and Statistics*. *Lecture Notes in Math.* **1869** 103–274. Springer, Berlin. MR2228384 https://doi.org/10.1007/11429579_2

[35] Funaki, T. and Spohn, H. (1997). Motion by mean curvature from the Ginzburg–Landau $\nabla\phi$ interface model. *Comm. Math. Phys.* **185** 1–36. MR1463032 https://doi.org/10.1007/s002200050080

[36] Fyodorov, Y., Hiary, G. and Keating, J. (2012). Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function. *Phys. Rev. Lett.* **108** 170601. https://doi.org/10.1103/PhysRevLett.108.170601

[37] Garban, C. and Sepúlveda, A. (2023). Quantitative bounds on vortex fluctuations in 2d Coulomb gas and maximum of the integer-valued Gaussian free field. *Proc. Lond. Math. Soc.* (3) **127** 653–708. MR4639954 https://doi.org/10.1112/plms.12551

[38] Giacomin, G., Olla, S. and Spohn, H. (2001). Equilibrium fluctuations for $\nabla\phi$ interface model. *Ann. Probab.* **29** 1138–1172. MR1872740 https://doi.org/10.1214/aop/1015345600

[39] HARPER, A. J. (2020). The Riemann zeta function in short intervals [after Najnudel, and Arguin, Belius, Bourgade, Radziwiłłand Soundararajan]. **1161**, 391–414. Astérisque, 422, Séminaire Bourbaki. Vol. 2018/2019. Exposés 1151–1165. MR4224641 https://doi.org/10.24033/ast

[40] HELFFER, B. and SJÖSTRAND, J. (1994). On the correlation for Kac-like models in the convex case. *J. Stat. Phys.* **74** 349–409. MR1257821 https://doi.org/10.1007/BF02186817

[41] HU, X., MILLER, J. and PERES, Y. (2010). Thick points of the Gaussian free field. *Ann. Probab.* **38** 896–926. MR2642894 https://doi.org/10.1214/09-AOP498

[42] JEGO, A. (2020). Thick points of random walk and the Gaussian free field. *Electron. J. Probab.* **25** Paper No. 32, 39. MR4073693 https://doi.org/10.1214/20-ejp433

[43] KHARASH, V. and PELED, R. (2017). The Fröhlich-Spencer proof of the Berezinskii–Kosterlitz–Thouless transition. Available at arXiv:1706.07737.

[44] LAMBERT, G. (2020). Maximum of the characteristic polynomial of the Ginibre ensemble. *Comm. Math. Phys.* **378** 943–985. MR4134939 https://doi.org/10.1007/s00220-020-03813-1

[45] LAMBERT, G. and PAQUETTE, E. (2019). The law of large numbers for the maximum of almost Gaussian log-correlated fields coming from random matrices. *Probab. Theory Related Fields* **173** 157–209. MR3916106 https://doi.org/10.1007/s00440-018-0832-2

[46] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk: A Modern Introduction. Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. MR2677157 https://doi.org/10.1017/CBO9780511750854

[47] LI, C. T. and ANANTHARAM, V. (2019). Pairwise multi-marginal optimal transport and embedding for Earth mover's distance. Available at arXiv:1908.01388.

[48] LINDVALL, T. (1992). *Lectures on the Coupling Method. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. Wiley, New York. MR1180522

[49] LUBETZKY, E., MARTINELLI, F. and SLY, A. (2016). Harmonic pinnacles in the discrete Gaussian model. *Comm. Math. Phys.* **344** 673–717. MR3508158 https://doi.org/10.1007/s00220-016-2628-5

[50] MALLER, R. A. (1978). A local limit theorem for independent random variables. *Stoch. Process. Appl.* **7** 101–111. MR0492695 https://doi.org/10.1016/0304-4149(78)90041-8

[51] MILLER, J. (2011). Fluctuations for the Ginzburg–Landau $\nabla\phi$ interface model on a bounded domain. *Comm. Math. Phys.* **308** 591–639. MR2855536 https://doi.org/10.1007/s00220-011-1315-9

[52] NADDAF, A. and SPENCER, T. (1997). On homogenization and scaling limit of some gradient perturbations of a massless free field. *Comm. Math. Phys.* **183** 55–84. MR1461951 https://doi.org/10.1007/BF02509796

[53] PAQUETTE, E. and ZEITOUNI, O. (2025). The extremal landscape for the CβE ensemble. *Forum Math. Sigma* **13** Paper No. e1, 126. MR4849370 https://doi.org/10.1017/fms.2024.129

[54] PERKINS, E. (1986). The Cereteli-Davis solution to the H^1-embedding problem and an optimal embedding in Brownian motion. In *Seminar on Stochastic Processes*, 1985 (*Gainesville*, *Fla.*, 1985). *Progr. Probab. Statist.* **12** 172–223. Birkhäuser, Boston, MA. MR0896743

[55] PRÉKOPA, A. (1973). On logarithmic concave measures and functions. *Acta Sci. Math.* (*Szeged*) **34** 335–343. MR0404557

[56] SCHWEIGER, F. (2020). The maximum of the four-dimensional membrane model. *Ann. Probab.* **48** 714–741. MR4089492 https://doi.org/10.1214/19-AOP1372

[57] SCHWEIGER, F. and ZEITOUNI, O. (2024). The maximum of log-correlated Gaussian fields in random environment. *Comm. Pure Appl. Math.* **77** 2778–2859. MR4720221 https://doi.org/10.1002/cpa.22181

[58] SHEFFIELD, S. (2005). Random surfaces. *Astérisque* **304** vi+175. MR2251117

[59] WIRTH, M. (2019). Maximum of the integer-valued Gaussian free field. Available at arXiv:1907.08868.

[60] WU, W. (2025). Local central limit theorem for gradient field models. *Probab. Theory Related Fields* **191** 1–40. MR4869252 https://doi.org/10.1007/s00440-024-01330-z

[61] WU, W. and ZEITOUNI, O. (2019). Subsequential tightness of the maximum of two dimensional Ginzburg–Landau fields. *Electron. Commun. Probab.* **24** Paper No. 19, 12. MR3933043 https://doi.org/10.1214/19-ECP215

[62] ZEITOUNI, O. (2016). Branching random walks and Gaussian fields. In *Probability and Statistical Physics in St. Petersburg. Proc. Sympos. Pure Math.* **91** 437–471. Amer. Math. Soc., Providence, RI. MR3526836 https://doi.org/10.1090/pspum/091/01544

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RENORMALIZATION OF A 1D QUADRATIC SCHRÖDINGER MODEL WITH ADDITIVE NOISE

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The study is devoted to the interpretation and well-posedness of the stochastic NLS model

$$(\imath\partial_t - \Delta)u = |u|^2 + \dot{B}, \quad u_0 = 0, t \in \mathbb{R}, x \in \mathbb{T},$$

where $\dot{B}$ stands for a space-time fractional noise with index $H = (H_0, H_1)$ in a subset of $(0, 1)^2$.

We first establish that in the situation where $0 < 2H_0 + H_1 \leq 2$, the equation cannot be interpreted in a (classical) functional sense.

Our investigations then focus on the rough regime corresponding to the condition $\frac{7}{4} < 2H_0 + H_1 \leq 2$. In this specific case, we exhibit an *explicit* renormalization procedure allowing to restore the (local) convergence of the approximated solutions.

We follow a pathwise-type approach emphasizing the distinction between the stochastic objects at the core of the dynamics and the general deterministic machinery.

REFERENCES

[1] BALAN, R. M. and CONUS, D. (2016). Intermittency for the wave and heat equations with fractional noise in time. *Ann. Probab.* **44** 1488–1534. MR3474476 https://doi.org/10.1214/15-AOP1005
[2] BANG, O., CHRISTIANSEN, P. L., IF, F., RASMUSSEN, K. O. and GAIDIDEI, Y. B. (1994). Temperature effects in a nonlinear model of monolayer Scheibe aggregates. *Phys. Rev. E* **49** 4627–4636.
[3] BARBU, V., RÖCKNER, M. and ZHANG, D. (2016). Stochastic nonlinear Schrödinger equations. *Nonlinear Anal.* **136** 168–194. MR3474409 https://doi.org/10.1016/j.na.2016.02.010
[4] BÉNYI, Á., OH, T. and POCOVNICU, O. (2015). Wiener randomization on unbounded domains and an application to almost sure well-posedness of NLS. In *Excursions in Harmonic Analysis*, *Vol.* 4. *Appl. Numer. Harmon. Anal.* 3–25. Birkhäuser, Cham. MR3411090
[5] BÉNYI, Á., OH, T. and POCOVNICU, O. (2015). On the probabilistic Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$, $d \geq 3$. *Trans. Amer. Math. Soc. Ser. B* **2** 1–50. MR3350022 https://doi.org/10.1090/btran/6
[6] BÉNYI, Á., OH, T. and POCOVNICU, O. (2019). Higher order expansions for the probabilistic local Cauchy theory of the cubic nonlinear Schrödinger equation on $\mathbb{R}^3$. *Trans. Amer. Math. Soc. Ser. B* **6** 114–160. MR3919013 https://doi.org/10.1090/btran/29
[7] BOURGAIN, J. (1993). Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. *Geom. Funct. Anal.* **3** 107–156. MR1209299 https://doi.org/10.1007/BF01896020
[8] BOURGAIN, J. (1994). Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.* **166** 1–26. MR1309539
[9] BOURGAIN, J. (1996). Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Comm. Math. Phys.* **176** 421–445. MR1374420
[10] BRZEŹNIAK, Z. and MILLET, A. (2013). On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. *Potential Anal.* **41** 269–315. MR3232027 https://doi.org/10.1007/s11118-013-9369-2
[11] BURQ, N., CAMPS, N., SUN, C. and TZVETKOV, N. Probabilistic well-posedeness for the nonlinear Schrödinger equation on the 2d sphere I: Positive regularities. Available at arXiv:2404.18229.

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[12] BURQ, N., THOMANN, L. and TZVETKOV, N. (2013). Long time dynamics for the one dimensional non linear Schrödinger equation. *Ann. Inst. Fourier* (*Grenoble*) **63** 2137–2198. MR3237443 https://doi.org/10.5802/aif.2825

[13] BURQ, N. and TZVETKOV, N. (2007). Invariant measure for a three dimensional nonlinear wave equation. *Int. Math. Res. Not. IMRN* **22** Art. ID rnm108, 26. MR2376217 https://doi.org/10.1093/imrn/rnm108

[14] BURQ, N. and TZVETKOV, N. (2008). Random data Cauchy theory for supercritical wave equations. I. Local theory. *Invent. Math.* **173** 449–475. MR2425133 https://doi.org/10.1007/s00222-008-0124-z

[15] BURQ, N. and TZVETKOV, N. (2008). Random data Cauchy theory for supercritical wave equations. II. A global existence result. *Invent. Math.* **173** 477–496. MR2425134 https://doi.org/10.1007/s00222-008-0123-0

[16] BURQ, N. and TZVETKOV, N. (2014). Probabilistic well-posedness for the cubic wave equation. *J. Eur. Math. Soc.* (*JEMS*) **16** 1–30. MR3141727 https://doi.org/10.4171/JEMS/426

[17] CHEUNG, K. and MOSINCAT, R. (2019). Stochastic nonlinear Schrödinger equations on tori. *Stoch. Partial Differ. Equ. Anal. Comput.* **7** 169–208. MR3949103 https://doi.org/10.1007/s40072-018-0125-x

[18] CHOUK, K. and GUBINELLI, M. (2015). Nonlinear PDEs with modulated dispersion I: Nonlinear Schrödinger equations. *Comm. Partial Differential Equations* **40** 2047–2081. MR3418825 https://doi.org/10.1080/03605302.2015.1073300

[19] COLLIANDER, J. and OH, T. (2012). Almost sure well-posedness of the cubic nonlinear Schrödinger equation below $L^2(\mathbb{T})$. *Duke Math. J.* **161** 367–414. MR2881226 https://doi.org/10.1215/00127094-1507400

[20] CONSTANTIN, P. and SAUT, J.-C. (1988). Local smoothing properties of dispersive equations. *J. Amer. Math. Soc.* **1** 413–439. MR0928265 https://doi.org/10.2307/1990923

[21] CONTI, C. (2012). Solitonization of the Anderson localization. *Phys. Rev. A* **86**.

[22] DA PRATO, G. and DEBUSSCHE, A. (2003). Strong solutions to the stochastic quantization equations. *Ann. Probab.* **31** 1900–1916. MR2016604 https://doi.org/10.1214/aop/1068646370

[23] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136 https://doi.org/10.1017/CBO9780511666223

[24] DE BOUARD, A. and DEBUSSCHE, A. (1999). A stochastic nonlinear Schrödinger equation with multiplicative noise. *Comm. Math. Phys.* **205** 161–181. MR1706888 https://doi.org/10.1007/s002200050672

[25] DE BOUARD, A. and DEBUSSCHE, A. (2002). On the effect of a noise on the solutions of the focusing supercritical nonlinear Schrödinger equation. *Probab. Theory Related Fields* **123** 76–96. MR1906438 https://doi.org/10.1007/s004400100183

[26] DE BOUARD, A. and DEBUSSCHE, A. (2003). The stochastic nonlinear Schrödinger equation in H^1. *Stoch. Anal. Appl.* **21** 97–126. MR1954077 https://doi.org/10.1081/SAP-120017534

[27] DE BOUARD, A. and DEBUSSCHE, A. (2010). The nonlinear Schrödinger equation with white noise dispersion. *J. Funct. Anal.* **259** 1300–1321. MR2652190 https://doi.org/10.1016/j.jfa.2010.04.002

[28] DE BOUARD, A. and FUKUIZUMI, R. (2012). Representation formula for stochastic Schrödinger evolution equations and applications. *Nonlinearity* **25** 2993–3022. MR2980868 https://doi.org/10.1088/0951-7715/25/11/2993

[29] DEBUSSCHE, A., LIU, R., TZVETKOV, N. and VISCIGLIA, N. (2024). Global well-posedness of the 2D nonlinear Schrödinger equation with multiplicative spatial white noise on the full space. *Probab. Theory Related Fields* **189** 1161–1218. MR4771113 https://doi.org/10.1007/s00440-024-01288-y

[30] DEBUSSCHE, A. and MARTIN, J. (2019). Solution to the stochastic Schrödinger equation on the full space. *Nonlinearity* **32** 1147–1174. MR3923163 https://doi.org/10.1088/1361-6544/aaf50e

[31] DEBUSSCHE, A. and TSUTSUMI, Y. (2011). 1D quintic nonlinear Schrödinger equation with white noise dispersion. *J. Math. Pures Appl.* (9) **96** 363–376. MR2832639 https://doi.org/10.1016/j.matpur.2011.02.002

[32] DEBUSSCHE, A. and WEBER, H. (2018). The Schrödinger equation with spatial white noise potential. *Electron. J. Probab.* **23** Paper No. 28, 16. MR3785398 https://doi.org/10.1214/18-EJP143

[33] DENG, Y. (2012). Two-dimensional nonlinear Schrödinger equation with random radial data. *Anal. PDE* **5** 913–960. MR3022846 https://doi.org/10.2140/apde.2012.5.913

[34] DENG, Y., NAHMOD, A. R. and YUE, H. (2021). Optimal local well-posedness for the periodic derivative nonlinear Schrödinger equation. *Comm. Math. Phys.* **384** 1061–1107. MR4259382 https://doi.org/10.1007/s00220-020-03898-8

[35] DENG, Y., NAHMOD, A. R. and YUE, H. (2022). Random tensors, propagation of randomness, and nonlinear dispersive equations. *Invent. Math.* **228** 539–686. MR4411729 https://doi.org/10.1007/s00222-021-01084-8

[36] DENG, Y., NAHMOD, A. R. and YUE, H. (2024). Invariant Gibbs measures and global strong solutions for nonlinear Schrödinger equations in dimension two. *Ann. of Math.* (2) **200** 399–486. MR4792067 https://doi.org/10.4007/annals.2024.200.2.1

[37] DENG, Y., TZVETKOV, N. and VISCIGLIA, N. (2015). Invariant measures and long time behaviour for the Benjamin–Ono equation III. *Comm. Math. Phys.* **339** 815–857. MR3385985 https://doi.org/10.1007/s00220-015-2431-8

[38] DEYA, A. On the 1d stochastic Schrödinger product. *Stoch. Partial Differ. Equ. Anal. Comput.* To appear. https://doi.org/10.1007/s40072-025-00381-0

[39] DEYA, A. (2016). On a modelled rough heat equation. *Probab. Theory Related Fields* **166** 1–65. MR3547736 https://doi.org/10.1007/s00440-015-0650-8

[40] DEYA, A. (2019). A nonlinear wave equation with fractional perturbation. *Ann. Probab.* **47** 1775–1810. MR3945759 https://doi.org/10.1214/18-AOP1296

[41] DEYA, A., SCHAEFFER, N. and THOMANN, L. (2021). A nonlinear Schrödinger equation with fractional noise. *Trans. Amer. Math. Soc.* **374** 4375–4422. MR4251233 https://doi.org/10.1090/tran/8368

[42] DUBOSCQ, R. and RÉVEILLAC, A. (2022). On a stochastic Hardy–Littlewood–Sobolev inequality with application to Strichartz estimates for a noisy dispersion. *Ann. Henri Lebesgue* **5** 263–274. MR4443290 https://doi.org/10.5802/ahl.122

[43] ERDOĞAN, M. B. and TZIRAKIS, N. (2016). *Dispersive Partial Differential Equations: Wellposedness and Applicationd. London Mathematical Society Student Texts* **86**. Cambridge Univ. Press, Cambridge. MR3559154 https://doi.org/10.1017/CBO9781316563267

[44] FORLANO, J., OH, T. and WANG, Y. (2020). Stochastic nonlinear Schrödinger equation with almost space-time white noise. *J. Aust. Math. Soc.* **109** 44–67. MR4120796 https://doi.org/10.1017/s1446788719000156

[45] FUJIWARA, K. and GEORGIEV, V. (2021). On global existence of L^2 solutions for 1D periodic NLS with quadratic nonlinearity. *J. Math. Phys.* **62** Paper No. 091504, 9. MR4308611 https://doi.org/10.1063/5.0033101

[46] GARNIER, J., ABDULLAEV, F. K. and BAIZAKOV, B. B. (2004). Collapse of a Bose–Einstein condensate induced by fluctuations of the laser intensity. *Phys. Rev. A* **69**.

[47] GHOFRANIHA, N., GENTILINI, S., FOLLI, V., DELRE, E. and CONTI, C. (2012). Shock waves in disordered media. *Phys. Rev. Lett.* **109**.

[48] GINIBRE, J. (1996). Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain). (French) [The Cauchy problem for periodic semilinear PDE in space variables (after Bourgain)] *Séminaire Bourbaki, Vol. 1994/95. Astérisque* No. 237 (1996), Exp. No. 796, 4, 163–187. MR1423623

[49] GINIBRE, J., TSUTSUMI, Y. and VELO, G. (1997). On the Cauchy problem for the Zakharov system. *J. Funct. Anal.* **151** 384–436. MR1491547 https://doi.org/10.1006/jfan.1997.3148

[50] GUBINELLI, M., KOCH, H. and OH, T. (2018). Renormalization of the two-dimensional stochastic nonlinear wave equations. *Trans. Amer. Math. Soc.* **370** 7335–7359. MR3841850 https://doi.org/10.1090/tran/7452

[51] GUBINELLI, M., KOCH, H. and OH, T. (2024). Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity. *J. Eur. Math. Soc.* (*JEMS*) **26** 817–874. MR4721025 https://doi.org/10.4171/jems/1294

[52] HAIRER, M. (2014). A theory of regularity structures. *Invent. Math.* **198** 269–504. MR3274562 https://doi.org/10.1007/s00222-014-0505-4

[53] HORNUNG, F. (2018). The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates. *J. Evol. Equ.* **18** 1085–1114. MR3859442 https://doi.org/10.1007/s00028-018-0433-7

[54] HU, Y., NUALART, D. and SONG, J. (2011). Feynman–Kac formula for heat equation driven by fractional white noise. *Ann. Probab.* **39** 291–326. MR2778803 https://doi.org/10.1214/10-AOP547

[55] KENIG, C. E., PONCE, G. and VEGA, L. (1996). Quadratic forms for the 1-D semilinear Schrödinger equation. *Trans. Amer. Math. Soc.* **348** 3323–3353. MR1357398 https://doi.org/10.1090/S0002-9947-96-01645-5

[56] KISHIMOTO, N. (2019). A remark on norm inflation for nonlinear Schrödinger equations. *Commun. Pure Appl. Anal.* **18** 1375–1402. MR3917712 https://doi.org/10.3934/cpaa.2019067

[57] LIU, R. (2023). On the probabilistic well-posedness of the two-dimensional periodic nonlinear Schrödinger equation with the quadratic nonlinearity $|u|^2$. *J. Math. Pures Appl.* (9) **171** 75–101. MR4545169 https://doi.org/10.1016/j.matpur.2022.12.010

[58] NOURDIN, I. (2012). *Selected Aspects of Fractional Brownian Motion. Bocconi & Springer Series* **4**. Springer, Milan. MR3076266 https://doi.org/10.1007/978-88-470-2823-4

[59] NUALART, D. (2006). *The Malliavin Calculus and Related Topics*, 2nd ed. *Probability and Its Applications* (*New York*). Springer, Berlin. MR2200233

[60] OH, T. and OKAMOTO, M. (2020). On the stochastic nonlinear Schrödinger equations at critical regularities. *Stoch. Partial Differ. Equ. Anal. Comput.* **8** 869–894. MR4174072 https://doi.org/10.1007/s40072-019-00163-5

[61] OH, T. and OKAMOTO, M. (2021). Comparing the stochastic nonlinear wave and heat equations: A case study. *Electron. J. Probab.* **26** Paper No. 9, 44. MR4216522 https://doi.org/10.1214/20-EJP575

[62] OH, T., OKAMOTO, M. and TOLOMEO, L. (2024). Focusing Φ_3^4-model with a Hartree-type nonlinearity. *Mem. Amer. Math. Soc.* **304** vi+143. MR4850409 https://doi.org/10.1090/memo/1529

[63] OH, T., OKAMOTO, M. and TOLOMEO, L. (2025). *Stochastic Quantization of the Φ_3^3-Model. Memoirs of the European Mathematical Society* **16**. EMS Press, Berlin. MR4867559 https://doi.org/10.4171/mems/16

[64] OH, T., POCOVNICU, O. and WANG, Y. (2020). On the stochastic nonlinear Schrödinger equations with non-smooth additive noise. *Kyoto J. Math.* **60** 1227–1243. MR4175807 https://doi.org/10.1215/21562261-2019-0060

[65] OH, T., ROBERT, T. and TZVETKOV, N. (2023). Stochastic nonlinear wave dynamics on compact surfaces. *Ann. Henri Lebesgue* **6** 161–223. MR4593608 https://doi.org/10.5802/ahl.163

[66] OH, T. and THOMANN, L. (2018). A pedestrian approach to the invariant Gibbs measures for the 2-*d* defocusing nonlinear Schrödinger equations. *Stoch. Partial Differ. Equ. Anal. Comput.* **6** 397–445. MR3844655 https://doi.org/10.1007/s40072-018-0112-2

[67] OH, T., WANG, Y. and ZINE, Y. (2022). Three-dimensional stochastic cubic nonlinear wave equation with almost space-time white noise. *Stoch. Partial Differ. Equ. Anal. Comput.* **10** 898–963. MR4491506 https://doi.org/10.1007/s40072-022-00237-x

[68] PINAUD, O. (2014). A note on stochastic Schrödinger equations with fractional multiplicative noise. *J. Differ. Equ.* **256** 1467–1491. MR3145763 https://doi.org/10.1016/j.jde.2013.11.003

[69] RUNST, T. and SICKEL, W. (1996). *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations. De Gruyter Series in Nonlinear Analysis and Applications* **3**. de Gruyter, Berlin. MR1419319 https://doi.org/10.1515/9783110812411

[70] SCHAEFFER, N. (2024). Multilinear smoothing and local well-posedness of a stochastic quadratic nonlinear Schrödinger equation. *J. Theoret. Probab.* **37** 160–208. MR4716316 https://doi.org/10.1007/s10959-023-01282-5

[71] THOMANN, L. (2009). Random data Cauchy problem for supercritical Schrödinger equations. *Ann. Inst. Henri Poincaré C, Anal. Non Linéaire* **26** 2385–2402. MR2569900 https://doi.org/10.1016/j.anihpc.2009.06.001

[72] TZVETKOV, N. (2006). Invariant measures for the nonlinear Schrödinger equation on the disc. *Dyn. Partial Differ. Equ.* **3** 111–160. MR2227040 https://doi.org/10.4310/DPDE.2006.v3.n2.a2

[73] TZVETKOV, N. (2008). Invariant measures for the defocusing nonlinear Schrödinger equation. *Ann. Inst. Fourier (Grenoble)* **58** 2543–2604. MR2498359 https://doi.org/10.5802/aif.2422

[74] TZVETKOV, N. and VISCIGLIA, N. (2014). Invariant measures and long-time behavior for the Benjamin–Ono equation. *Int. Math. Res. Not. IMRN* **17** 4679–4714. MR3257548 https://doi.org/10.1093/imrn/rnt094

[75] TZVETKOV, N. and VISCIGLIA, N. (2015). Invariant measures and long time behaviour for the Benjamin–Ono equation II. *J. Math. Pures Appl.* (9) **103** 102–141. MR3281949 https://doi.org/10.1016/j.matpur.2014.03.009

[76] TZVETKOV, N. and VISCIGLIA, N. (2023). Two dimensional nonlinear Schrödinger equation with spatial white noise potential and fourth order nonlinearity. *Stoch. Partial Differ. Equ. Anal. Comput.* **11** 948–987. MR4624131 https://doi.org/10.1007/s40072-022-00251-z

[77] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour, XIV—1984. Lecture Notes in Math.* **1180** 265–439. Springer, Berlin. MR0876085 https://doi.org/10.1007/BFb0074920

[78] ZHANG, D. (2022). Strichartz and local smoothing estimates for stochastic dispersive equations with linear multiplicative noise. *SIAM J. Math. Anal.* **54** 5981–6017. MR4508067 https://doi.org/10.1137/21M1426304

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REFLECTING POISSON WALKS AND DYNAMICAL UNIVERSALITY IN p-ADIC RANDOM MATRIX THEORY

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We prove dynamical local limits for the singular numbers of p-adic random matrix products at both the bulk and edge. The limit object which we construct, the *reflecting Poisson sea*, may thus be viewed as a p-adic analogue of line ensembles appearing in classical random matrix theory. However, in contrast to those it is a discrete space Poisson-type particle system with only local reflection interactions and no obvious determinantal structure. The limits hold for any $\mathrm{GL}_n(\mathbb{Z}_p)$-invariant matrix distributions under weak universality hypotheses, with no spatial rescaling.

REFERENCES

[1] AHN, A. (2022). Airy point process via supersymmetric lifts. *Probab. Math. Phys.* **3** 869–938. MR4552230 https://doi.org/10.2140/pmp.2022.3.869

[2] AHN, A. (2023). Extremal singular values of random matrix products and Brownian motion on $\mathrm{GL}(N, \mathbb{C})$. *Probab. Theory Related Fields* **187** 949–997. MR4664588 https://doi.org/10.1007/s00440-023-01217-5

[3] AKEMANN, G., BURDA, Z. and KIEBURG, M. (2019). From integrable to chaotic systems: Universal local statistics of Lyapunov exponents. *Europhys. Lett.* **126** 40001.

[4] BARYSHNIKOV, Y. (2001). GUEs and queues. *Probab. Theory Related Fields* **119** 256–274. MR1818248 https://doi.org/10.1007/PL00008760

[5] BELLMAN, R. (1954). Limit theorems for non-commutative operations. I. *Duke Math. J.* **21** 491–500. MR0062368

[6] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York. MR0233396

[7] BORODIN, A. and FERRARI, P. L. (2008). Large time asymptotics of growth models on space-like paths. I. PushASEP. *Electron. J. Probab.* **13** 1380–1418. MR2438811 https://doi.org/10.1214/EJP.v13-541

[8] BORODIN, A. and GORIN, V. (2013). Markov processes of infinitely many nonintersecting random walks. *Probab. Theory Related Fields* **155** 935–997. MR3034797 https://doi.org/10.1007/s00440-012-0417-4

[9] BORODIN, A., GORIN, V. and STRAHOV, E. (2020). Product matrix processes as limits of random plane partitions. *Int. Math. Res. Not. IMRN* **20** 6713–6768. MR4172668 https://doi.org/10.1093/imrn/rny297

[10] CHEONG, G. and HUANG, Y. (2021). Cohen-Lenstra distributions via random matrices over complete discrete valuation rings with finite residue fields. *Illinois J. Math.* **65** 385–415. MR4278684 https://doi.org/10.1215/00192082-8939615

[11] CHEONG, G. and KAPLAN, N. (2022). Generalizations of results of Friedman and Washington on cokernels of random p-adic matrices. *J. Algebra* **604** 636–663. MR4416726 https://doi.org/10.1016/j.jalgebra.2022.03.035

[12] CHEONG, G. and YU, M. (2023). The cokernel of a polynomial of a random integral matrix. arXiv Preprint. Available at arXiv:2303.09125.

[13] CLANCY, J., KAPLAN, N., LEAKE, T., PAYNE, S. and WOOD, M. M. (2015). On a Cohen-Lenstra heuristic for Jacobians of random graphs. *J. Algebraic Combin.* **42** 701–723. MR3403177 https://doi.org/10.1007/s10801-015-0598-x

[14] COHEN, H. and LENSTRA, H. W. JR. (1984). Heuristics on class groups of number fields. In *Number Theory, Noordwijkerhout* 1983 (*Noordwijkerhout*, 1983). *Lecture Notes in Math.* **1068** 33–62. Springer, Berlin. MR0756082 https://doi.org/10.1007/BFb0099440

[15] COHN, H., ELKIES, N. and PROPP, J. (1996). Local statistics for random domino tilings of the Aztec diamond. *Duke Math. J.* **85** 117–166. MR1412441 https://doi.org/10.1215/S0012-7094-96-08506-3

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[16] CORWIN, I. and HAMMOND, A. (2014). Brownian Gibbs property for Airy line ensembles. *Invent. Math.* **195** 441–508. MR3152753 https://doi.org/10.1007/s00222-013-0462-3

[17] CORWIN, I. and HAMMOND, A. (2016). KPZ line ensemble. *Probab. Theory Related Fields* **166** 67–185. MR3547737 https://doi.org/10.1007/s00440-015-0651-7

[18] CRISANTI, A., PALADIN, G. and VULPIANI, A. (1993). *Products of Random Matrices in Statistical Physics*. *Springer Series in Solid-State Sciences* **104**. Springer, Berlin. MR1278483 https://doi.org/10.1007/978-3-642-84942-8

[19] DAUVERGNE, D., NICA, M. and VIRÁG, B. (2023). Uniform convergence to the Airy line ensemble. *Ann. Inst. Henri Poincaré Probab. Stat.* **59** 2220–2256. MR4663521 https://doi.org/10.1214/22-aihp1314

[20] DYSON, F. J. (1962). A Brownian-motion model for the eigenvalues of a random matrix. *J. Math. Phys.* **3** 1191–1198. MR0148397 https://doi.org/10.1063/1.1703862

[21] EVANS, S. N. (2002). Elementary divisors and determinants of random matrices over a local field. *Stoch. Process. Appl.* **102** 89–102. MR1934156 https://doi.org/10.1016/S0304-4149(02)00187-4

[22] FERRARI, P. L. and PRÄHOFER, M. (2006). One-dimensional stochastic growth and Gaussian ensembles of random matrices. *Markov Process. Related Fields* **12** 203–234. MR2249629

[23] FORRESTER, P. J. and LIU, D.-Z. (2016). Singular values for products of complex Ginibre matrices with a source: Hard edge limit and phase transition. *Comm. Math. Phys.* **344** 333–368. MR3493145 https://doi.org/10.1007/s00220-015-2507-5

[24] FRIEDMAN, E. and WASHINGTON, L. C. (1989). On the distribution of divisor class groups of curves over a finite field. In *Théorie des Nombres* (*Quebec*, *PQ*, 1987) 227–239. de Gruyter, Berlin. MR1024565

[25] FULTON, W. (2000). Eigenvalues, invariant factors, highest weights, and Schubert calculus. *Bull. Amer. Math. Soc.* (*N.S.*) **37** 209–249. MR1754641 https://doi.org/10.1090/S0273-0979-00-00865-X

[26] FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Statist.* **31** 457–469. MR0121828 https://doi.org/10.1214/aoms/1177705909

[27] GASPER, G. and RAHMAN, M. (2004). *Basic Hypergeometric Series*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **96**. Cambridge Univ. Press, Cambridge. MR2128719 https://doi.org/10.1017/CBO9780511526251

[28] GOL'DSHEĬD, I. Y. and MARGULIS, G. A. (1989). Lyapunov exponents of a product of random matrices. *RuMaS* **44** 11–71.

[29] GORIN, V. and KLEPTSYN, V. (2024). Universal objects of the infinite beta random matrix theory. *J. Eur. Math. Soc.* (*JEMS*) **26** 3429–3496. MR4767497 https://doi.org/10.4171/jems/1336

[30] GORIN, V. and SUN, Y. (2022). Gaussian fluctuations for products of random matrices. *Amer. J. Math.* **144** 287–393. MR4401507 https://doi.org/10.1353/ajm.2022.0006

[31] HUANG, J. (2022). Eigenvalues for the minors of Wigner matrices. *Ann. Inst. Henri Poincaré Probab. Stat.* **58** 2201–2215. MR4492976 https://doi.org/10.1214/21-aihp1226

[32] JOHANSSON, K. (2002). Non-intersecting paths, random tilings and random matrices. *Probab. Theory Related Fields* **123** 225–280. MR1900323 https://doi.org/10.1007/s004400100187

[33] JOHANSSON, K. and NORDENSTAM, E. (2006). Eigenvalues of GUE minors. *Electron. J. Probab.* **11** 1342–1371. MR2268547 https://doi.org/10.1214/EJP.v11-370

[34] KAHLE, M. (2014). Topology of random simplicial complexes: A survey. *AMS Contemp. Math* **620** 201–222.

[35] KAHLE, M., LUTZ, F. H., NEWMAN, A. and PARSONS, K. (2020). Cohen-Lenstra heuristics for torsion in homology of random complexes. *Exp. Math.* **29** 347–359. MR4134833 https://doi.org/10.1080/10586458.2018.1473821

[36] KATORI, M. and TANEMURA, H. (2010). Non-equilibrium dynamics of Dyson's model with an infinite number of particles. *Comm. Math. Phys.* **293** 469–497. MR2563791 https://doi.org/10.1007/s00220-009-0912-3

[37] KIPNIS, C. and LANDIM, C. (1999). *Scaling Limits of Interacting Particle Systems*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **320**. Springer, Berlin. MR1707314 https://doi.org/10.1007/978-3-662-03752-2

[38] LIU, D.-Z., WANG, D. and WANG, Y. (2023). Lyapunov exponent, universality and phase transition for products of random matrices. *Comm. Math. Phys.* **399** 1811–1855. MR4580535 https://doi.org/10.1007/s00220-022-04584-7

[39] MÉSZÁROS, A. (2020). The distribution of sandpile groups of random regular graphs. *Trans. Amer. Math. Soc.* **373** 6529–6594. MR4155185 https://doi.org/10.1090/tran/8127

[40] NGUYEN, H. H. and VAN PESKI, R. (2024). Universality for cokernels of random matrix products. *Adv. Math.* **438** Paper No. 109451. MR4676598 https://doi.org/10.1016/j.aim.2023.109451

[41] NGUYEN, H. H. and WOOD, M. M. (2025). Local and global universality of random matrix cokernels. *Math. Ann.* **391** 5117–5210. MR4884544 https://doi.org/10.1007/s00208-024-03050-0

[42] OSADA, H. (2012). Infinite-dimensional stochastic differential equations related to random matrices. *Probab. Theory Related Fields* **153** 471–509. MR2948684 https://doi.org/10.1007/s00440-011-0352-9

[43] OSADA, H. (2013). Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials. *Ann. Probab.* **41** 1–49. MR3059192 https://doi.org/10.1214/11-AOP736

[44] OSADA, H. and TANEMURA, H. (2020). Infinite-dimensional stochastic differential equations and tail σ-fields. *Probab. Theory Related Fields* **177** 1137–1242. MR4126938 https://doi.org/10.1007/s00440-020-00981-y

[45] SHEFFIELD, S. (2005). Random surfaces. *Astérisque* **304** vi+175. MR2251117

[46] SPOHN, H. (1987). Interacting Brownian particles: A study of Dyson's model. In *Hydrodynamic Behavior and Interacting Particle Systems* (*Minneapolis*, *Minn*., 1986). *IMA Vol. Math. Appl.* **9** 151–179. Springer, New York. MR0914993 https://doi.org/10.1007/978-1-4684-6347-7_13

[47] TSAI, L.-C. (2016). Infinite dimensional stochastic differential equations for Dyson's model. *Probab. Theory Related Fields* **166** 801–850. MR3568040 https://doi.org/10.1007/s00440-015-0672-2

[48] VAN PESKI, R. (2021). Limits and fluctuations of p-adic random matrix products. *Selecta Math.* (*N.S.*) **27** Paper No. 98. MR4323327 https://doi.org/10.1007/s00029-021-00709-3

[49] VAN PESKI, R. (2022). q-TASEP with position-dependent slowing. *Electron. J. Probab.* **27** Paper No. 151. MR4515707 https://doi.org/10.1214/22-ejp876

[50] VAN PESKI, R. (2024). Local limits in p-adic random matrix theory. *Proc. Lond. Math. Soc.* (3) **129** Paper No. e12626. MR4793279 https://doi.org/10.1112/plms.12626

[51] VAN PESKI, R. (2026). What is a p-adic Dyson Brownian motion? *Ann. Inst. Henri Poincaré Probab. Stat.* **62** 245–262. MR5036530 https://doi.org/10.1214/24-aihp1512

[52] VAN PESKI, R. (2023). Asymptotics, exact results, and analogies in p-adic random matrix theory. PhD thesis, Massachusetts Institute of Technology.

[53] WOOD, M. M. (2017). The distribution of sandpile groups of random graphs. *J. Amer. Math. Soc.* **30** 915–958. MR3671933 https://doi.org/10.1090/jams/866

[54] WOOD, M. M. (2018). Cohen–Lenstra heuristics and local conditions. *Res. Number Theory* **4** Paper No. 41. MR3858473 https://doi.org/10.1007/s40993-018-0134-x

[55] WOOD, M. M. (2019). Random integral matrices and the Cohen-Lenstra heuristics. *Amer. J. Math.* **141** 383–398. MR3928040 https://doi.org/10.1353/ajm.2019.0008

[56] WOOD, M. M. (2023). Probability theory for random groups arising in number theory. In *ICM—International Congress of Mathematicians. Vol.* 6. *Sections* 12–14 4476–4508. EMS Press, Berlin. MR4680411

[57] WU, X. (2023). The Bessel line ensemble. *Electron. J. Probab.* **28** Paper No. 77. MR4605337 https://doi.org/10.1214/23-ejp963

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FLUCTUATIONS OF SCHENSTED ROW INSERTION

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We investigate asymptotic probabilistic phenomena arising from the application of the Schensted row insertion algorithm, a key component of the Robinson–Schensted–Knuth (RSK) correspondence, to random inputs. Our analysis centers on a random tableau T with a given shape λ, which may itself be random or deterministic. We examine the stochastic properties of the position of the new box created when inserting a deterministic entry into T. Specifically, we focus on the fluctuations of this position around its expected value as the size of the Young diagram λ approaches infinity. Our findings reveal that these fluctuations are asymptotically Gaussian, with the mean and variance expressed in terms of Kerov's transition measure of the diagram λ. An important application of this analysis is the RSK algorithm applied to a finite, long sequence of independent, identically distributed random variables. While there remains a gap in the reasoning for this case, we present an explicit conjecture regarding its behavior.

REFERENCES

[1] BAIK, J., DEIFT, P. and JOHANSSON, K. (1999). On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* **12** 1119–1178. MR1682248 https://doi.org/10.1090/S0894-0347-99-00307-0

[2] BEN AROUS, G. and GUIONNET, A. (1997). Large deviations for Wigner's law and Voiculescu's noncommutative entropy. *Probab. Theory Related Fields* **108** 517–542. MR1465640 https://doi.org/10.1007/s004400050119

[3] BIANE, P. (1998). Representations of symmetric groups and free probability. *Adv. Math.* **138** 126–181. MR1644993 https://doi.org/10.1006/aima.1998.1745

[4] BIANE, P. (2001). Approximate factorization and concentration for characters of symmetric groups. *Int. Math. Res. Not.* **4** 179–192. MR1813797 https://doi.org/10.1155/S1073792801000113

[5] BOGACHEV, L. V. and SU, Z. (2007). Gaussian fluctuations of Young diagrams under the Plancherel measure. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463** 1069–1080. MR2310137 https://doi.org/10.1098/rspa.2006.1808

[6] DOŁĘGA, M., FÉRAY, V. and ŚNIADY, P. (2010). Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations. *Adv. Math.* **225** 81–120. MR2669350 https://doi.org/10.1016/j.aim.2010.02.011

[7] FERRARI, P. A., MARTIN, J. B. and PIMENTEL, L. P. R. (2009). A phase transition for competition interfaces. *Ann. Appl. Probab.* **19** 281–317. MR2498679 https://doi.org/10.1214/08-AAP542

[8] FULTON, W. (1997). *Young Tableaux: With Applications to Representation Theory and Geometry. London Mathematical Society Student Texts* **35**. Cambridge Univ. Press, Cambridge. MR1464693

[9] GORIN, V. and RAHMAN, M. (2019). Random sorting networks: Local statistics via random matrix laws. *Probab. Theory Related Fields* **175** 45–96. MR4009705 https://doi.org/10.1007/s00440-018-0886-1

[10] GUSTAVSSON, J. (2005). Gaussian fluctuations of eigenvalues in the GUE. *Ann. Inst. Henri Poincaré Probab. Stat.* **41** 151–178. MR2124079 https://doi.org/10.1016/j.anihpb.2004.04.002

[11] KENDALL, M. G. and STUART, A. (1979). *The Advanced Theory of Statistics*. (*In Three Volumes*) *Vol*. 2: *Inference and Relationship*. 4th ed. Charles Griffin & Co., Ltd., London, High Wycombe. 748 p. £ 23.50.

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[12] KEROV, S. (1993). Gaussian limit for the Plancherel measure of the symmetric group. *C. R. Acad. Sci. Paris Sér. I Math.* **316** 303–308. MR1204294

[13] KEROV, S. V. (1993). Transition probabilities of continual Young diagrams and the Markov moment problem. *Funct. Anal. Appl.* **27** 104–117. MR1251166 https://doi.org/10.1007/BF01085981

[14] KEROV, S. V. (2003). *Asymptotic Representation Theory of the Symmetric Group and Its Applications in Analysis*. *Translations of Mathematical Monographs* **219**. Amer. Math. Soc., Providence, RI. Translated from the Russian manuscript by N. V. Tsilevich. With a foreword by A. Vershik and comments by G. Olshanski. MR1984868 https://doi.org/10.1090/mmono/219

[15] KEROV, S. V. and VERSHIK, A. M. (1986). The characters of the infinite symmetric group and probability properties of the Robinson–Schensted–Knuth algorithm. *SIAM J. Algebr. Discrete Methods* **7** 116–124. MR0819713 https://doi.org/10.1137/0607014

[16] LOGAN, B. F. and SHEPP, L. A. (1977). A variational problem for random Young tableaux. *Adv. Math.* **26** 206–222. MR1417317 https://doi.org/10.1016/0001-8708(77)90030-5

[17] MARCINIAK, M., MAŚLANKA, Ł. and ŚNIADY, P. (2021). Poisson limit of bumping routes in the Robinson–Schensted correspondence. *Probab. Theory Related Fields* **181** 1053–1103. MR4344138 https://doi.org/10.1007/s00440-021-01084-y

[18] MARCINIAK, M. and ŚNIADY, P. (2024). Cumulants of threshold for Schensted row insertion into random tableaux. Preprint. Available at arXiv:2407.06213. https://doi.org/10.48550/arXiv

[19] MINGO, J. A. and SPEICHER, R. (2017). *Free Probability and Random Matrices*. *Fields Institute Monographs* **35**. Springer, New York. MR3585560 https://doi.org/10.1007/978-1-4939-6942-5

[20] OKOUNKOV, A. (2000). Random matrices and random permutations. *Int. Math. Res. Not.* **20** 1043–1095. MR1802530 https://doi.org/10.1155/S1073792800000532

[21] PITTEL, B. and ROMIK, D. (2007). Limit shapes for random square Young tableaux. *Adv. in Appl. Math.* **38** 164–209. MR2290809 https://doi.org/10.1016/j.aam.2005.12.005

[22] RAHMAN, M. and VIRÁG, B. (2025). Infinite geodesics, competition interfaces and the second class particle in the scaling limit. *Ann. Inst. Henri Poincaré Probab. Stat.* **61** 1075–1126. MR4901636 https://doi.org/10.1214/23-aihp1454

[23] ROMIK, D. (2015). *The Surprising Mathematics of Longest Increasing Subsequences*. *Institute of Mathematical Statistics Textbooks* **4**. Cambridge Univ. Press, New York. MR3468738

[24] ROMIK, D. and ŚNIADY, P. (2015). Jeu de taquin dynamics on infinite Young tableaux and second class particles. *Ann. Probab.* **43** 682–737. MR3306003 https://doi.org/10.1214/13-AOP873

[25] ROST, H. (1981). Nonequilibrium behaviour of a many particle process: Density profile and local equilibria. *Z. Wahrsch. Verw. Gebiete* **58** 41–53. MR0635270 https://doi.org/10.1007/BF00536194

[26] ŚNIADY, P. (2006). Gaussian fluctuations of characters of symmetric groups and of Young diagrams. *Probab. Theory Related Fields* **136** 263–297. MR2240789 https://doi.org/10.1007/s00440-005-0483-y

[27] ŚNIADY, P. (2025). Modulus of continuity of Kerov transition measure for continual Young diagrams. *Electron. J. Probab.* **30** Paper No. 108, 26. MR4926018 https://doi.org/10.1214/25-ejp1369

[28] STANLEY, R. P. (1999). *Enumerative Combinatorics*, *Vol.* 2. *Cambridge Studies in Advanced Mathematics* **62**. Cambridge Univ. Press, Cambridge. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. MR1676282 https://doi.org/10.1017/CBO9780511609589

[29] THOMA, E. (1964). Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. *Math. Z.* **85** 40–61. MR0173169 https://doi.org/10.1007/BF01114877

[30] TRACY, C. A. and WIDOM, H. (1994). Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.* **159** 151–174. MR1257246

[31] VASSILIEV, N. N., DUZHIN, V. S. and KUZMIN, A. D. (2019). Investigation of properties of equivalence classes of permutations by inverse Robinson–Schensted–Knuth transformation. *Inf. Control Syst.* **98** 11–22. https://doi.org/10.31799/1684-8853-2019-1-11-22

[32] VERSHIK, A. M. and KEROV, S. V. (1981). Asymptotic theory of the characters of a symmetric group. *Funktsional. Anal. i Prilozhen.* **15** 15–27, 96. MR0639197

[33] VERSHIK, A. M. and KEROV, S. V. (1977). Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux. *Sov. Math., Dokl.* **18** 527–531. MR0480398

[34] VOICULESCU, D. (1994). The analogues of entropy and of Fisher's information measure in free probability theory. II. *Invent. Math.* **118** 411–440. MR1296352 https://doi.org/10.1007/BF01231539

[35] WOJTYNIAK, K. (2019). Asymptotyka losowych trajektorii jeu de taquin, Bachelor's Thesis. Nicolaus Copernicus Univ. in Toruń.

[36] LAURITZEN, S. L. and HALD, A. (2002). *Thiele*: *Pioneer in Statistics*. Oxford Univ. Press, New York. Thiele's papers translated from the Danish by Steffen L. Lauritzen. With appreciations of Thiele's work by Lauritzen and A. Hald. MR2055773 https://doi.org/10.1093/acprof:oso/9780198509721.001.0001

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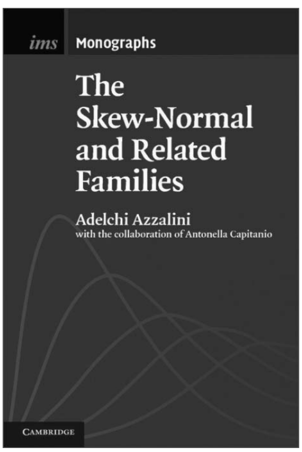

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