ISSN 0090-5364 (print)
ISSN 2168-8966 (online)

THE ANNALS *of* STATISTICS

AN OFFICIAL JOURNAL OF THE
INSTITUTE OF MATHEMATICAL STATISTICS

Articles

Vol. 46, No. 2—April 2018

THE ANNALS OF STATISTICS Vol. 46, No. 2, pp. 457–923 April 2018

INSTITUTE OF MATHEMATICAL STATISTICS

(Organized September 12, 1935)

The purpose of the Institute is to foster the development and dissemination of the theory and applications of statistics and probability.

The Annals of Statistics **[ISSN 0090-5364 (print); ISSN 2168-8966 (online)],** Volume 46, Number 2, April 2018. Published bimonthly by the Institute of Mathematical Statistics, 3163 Somerset Drive, Cleveland, Ohio 44122, USA. Periodicals postage paid at Cleveland, Ohio, and at additional mailing offices.

POSTMASTER: Send address changes to *The Annals of Statistics*, Institute of Mathematical Statistics, Dues and Subscriptions Office, 9650 Rockville Pike, Suite L 2310, Bethesda, Maryland 20814-3998, USA.

The Annals of Statistics
2018, Vol. 46, No. 2, 457–468
https://doi.org/10.1214/17-AOS1555

STRONG ORTHOGONAL ARRAYS OF STRENGTH TWO PLUS

BY YUANZHEN HE, CHING-SHUI CHENG AND BOXIN TANG[1]

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Strong orthogonal arrays were recently introduced and studied in He and Tang [*Biometrika* **100** (2013) 254–260] as a class of space-filling designs for computer experiments. To enjoy the benefits of better space-filling properties, when compared to ordinary orthogonal arrays, strong orthogonal arrays need to have strength three or higher, which may require run sizes that are too large for experimenters to afford. To address this problem, we introduce a new class of arrays, called strong orthogonal arrays of strength two plus. These arrays, while being more economical than strong orthogonal arrays of strength three, still enjoy the better two-dimensional space-filling property of the latter. Among the many results we have obtained on the characterizations and constructions of strong orthogonal arrays of strength two plus, worth special mention is their intimate connection with second-order saturated designs.

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MSC2010 subject classifications. Primary 62K15; secondary 05B15.

Key words and phrases. Complementary design, computer experiment, Latin hypercube, second-order saturated design, space-filling design.

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The Annals of Statistics
2018, Vol. 46, No. 2, 469–499
https://doi.org/10.1214/17-AOS1556

STATISTICAL INFERENCE FOR SPATIAL STATISTICS DEFINED IN THE FOURIER DOMAIN

BY SUHASINI SUBBA RAO[1]

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A class of Fourier based statistics for irregular spaced spatial data is introduced. Examples include the Whittle likelihood, a parametric estimator of the covariance function based on the L_2-contrast function and a simple nonparametric estimator of the spatial autocovariance which is a nonnegative function. The Fourier based statistic is a quadratic form of a discrete Fourier-type transform of the spatial data. Evaluation of the statistic is computationally tractable, requiring $O(nb)$ operations, where b are the number of Fourier frequencies used in the definition of the statistic and n is the sample size. The asymptotic sampling properties of the statistic are derived using both increasing domain and fixed-domain spatial asymptotics. These results are used to construct a statistic which is asymptotically pivotal.

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Key words and phrases. Fixed and increasing domain asymptotics, irregular spaced locations, quadratic forms, spatial spectral density function, stationary spatial random fields.

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The Annals of Statistics
2018, Vol. 46, No. 2, 500–525
https://doi.org/10.1214/17-AOS1558

ON THE INFERENCE ABOUT THE SPECTRAL DISTRIBUTION OF HIGH-DIMENSIONAL COVARIANCE MATRIX BASED ON HIGH-FREQUENCY NOISY OBSERVATIONS

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In practice, observations are often contaminated by noise, making the resulting sample covariance matrix a signal-plus-noise sample covariance matrix. Aiming to make inferences about the spectral distribution of the population covariance matrix under such a situation, we establish an asymptotic relationship that describes how the limiting spectral distribution of (signal) sample covariance matrices depends on that of signal-plus-noise-type sample covariance matrices. As an application, we consider inferences about the spectral distribution of integrated covolatility (ICV) matrices of high-dimensional diffusion processes based on high-frequency data with microstructure noise. The (slightly modified) pre-averaging estimator is a signal-plus-noise sample covariance matrix, and the aforementioned result, together with a (generalized) connection between the spectral distribution of signal sample covariance matrices and that of the population covariance matrix, enables us to propose a two-step procedure to consistently estimate the spectral distribution of ICV for a class of diffusion processes. An alternative approach is further proposed, which possesses several desirable properties: it is more robust, it eliminates the effects of microstructure noise, and the asymptotic relationship that enables consistent estimation of the spectral distribution of ICV is the standard Marčenko–Pastur equation. The performance of the two approaches is examined via simulation studies under both synchronous and asynchronous observation settings.

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MSC2010 subject classifications. Primary 62H12; secondary 62G99, 60F15.

Key words and phrases. High-dimension, high-frequency, integrated covariance matrices, Marčenko–Pastur equation, microstructure noise.

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The Annals of Statistics
2018, Vol. 46, No. 2, 526–554
https://doi.org/10.1214/17-AOS1559

ONLINE RULES FOR CONTROL OF FALSE DISCOVERY RATE AND FALSE DISCOVERY EXCEEDANCE

BY ADEL JAVANMARD[1] AND ANDREA MONTANARI[2]

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Multiple hypothesis testing is a core problem in statistical inference and arises in almost every scientific field. Given a set of null hypotheses $\mathcal{H}(n) = (H_1, \dots, H_n)$, Benjamini and Hochberg [*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **57** (1995) 289–300] introduced the false discovery rate (FDR), which is the expected proportion of false positives among rejected null hypotheses, and proposed a testing procedure that controls FDR below a preassigned significance level. Nowadays FDR is the criterion of choice for large-scale multiple hypothesis testing.

In this paper we consider the problem of controlling FDR in an *online manner*. Concretely, we consider an ordered—possibly infinite—sequence of null hypotheses $\mathcal{H} = (H_1, H_2, H_3, \dots)$ where, at each step i, the statistician must decide whether to reject hypothesis H_i having access only to the previous decisions. This model was introduced by Foster and Stine [*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **70** (2008) 429–444].

We study a class of *generalized alpha investing* procedures, first introduced by Aharoni and Rosset [*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **76** (2014) 771–794]. We prove that any rule in this class controls online FDR, provided p-values corresponding to true nulls are independent from the other p-values. Earlier work only established mFDR control. Next, we obtain conditions under which generalized alpha investing controls FDR in the presence of general p-values dependencies. We also develop a modified set of procedures that allow to control the false discovery exceedance (the tail of the proportion of false discoveries). Finally, we evaluate the performance of online procedures on both synthetic and real data, comparing them with offline approaches, such as adaptive Benjamini–Hochberg.

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MSC2010 subject classifications. Primary 62F03, 62F05; secondary 62L99.

Key words and phrases. Hypothesis testing, false discovery rate (FDR), false discovery exceedance (FDX), online decision making.

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The Annals of Statistics
2018, Vol. 46, No. 2, 555–579
https://doi.org/10.1214/17-AOS1560

FREQUENCY DOMAIN MINIMUM DISTANCE INFERENCE FOR POSSIBLY NONINVERTIBLE AND NONCAUSAL ARMA MODELS

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This article introduces frequency domain minimum distance procedures for performing inference in general, possibly non causal and/or noninvertible, autoregressive moving average (ARMA) models. We use information from higher order moments to achieve identification on the location of the roots of the AR and MA polynomials for non-Gaussian time series. We propose a minimum distance estimator that optimally combines the information contained in second, third, and fourth moments. Contrary to existing estimators, the proposed one is consistent under general assumptions, and may improve on the efficiency of estimators based on only second order moments. Our procedures are also applicable for processes for which either the third or the fourth order spectral density is the zero function.

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MSC2010 subject classifications. 62M10, 62F12.

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The Annals of Statistics
2018, Vol. 46, No. 2, 580–610
https://doi.org/10.1214/17-AOS1561

ON CONSISTENCY AND SPARSITY FOR SLICED INVERSE REGRESSION IN HIGH DIMENSIONS

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We provide here a framework to analyze the phase transition phenomenon of slice inverse regression (SIR), a supervised dimension reduction technique introduced by Li [*J. Amer. Statist. Assoc.* **86** (1991) 316–342]. Under mild conditions, the asymptotic ratio $\rho = \lim p/n$ is the phase transition parameter and the SIR estimator is consistent if and only if $\rho = 0$. When dimension p is greater than n, we propose a diagonal thresholding screening SIR (DT-SIR) algorithm. This method provides us with an estimate of the eigenspace of $\mathrm{var}(\mathbb{E}[\boldsymbol{x}|y])$, the covariance matrix of the conditional expectation. The desired dimension reduction space is then obtained by multiplying the inverse of the covariance matrix on the eigenspace. Under certain sparsity assumptions on both the covariance matrix of predictors and the loadings of the directions, we prove the consistency of DT-SIR in estimating the dimension reduction space in high-dimensional data analysis. Extensive numerical experiments demonstrate superior performances of the proposed method in comparison to its competitors.

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The Annals of Statistics
2018, Vol. 46, No. 2, 611–641
https://doi.org/10.1214/17-AOS1562

REGULARIZATION AND THE SMALL-BALL METHOD I: SPARSE RECOVERY

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We obtain bounds on estimation error rates for regularization procedures of the form

$$\hat{f} \in \underset{f \in F}{\operatorname{argmin}} \left(\frac{1}{N} \sum_{i=1}^{N} (Y_i - f(X_i))^2 + \lambda \Psi(f) \right)$$

when Ψ is a norm and F is convex.

Our approach gives a common framework that may be used in the analysis of learning problems and regularization problems alike. In particular, it sheds some light on the role various notions of sparsity have in regularization and on their connection with the size of subdifferentials of Ψ in a neighborhood of the true minimizer.

As "proof of concept" we extend the known estimates for the LASSO, SLOPE and trace norm regularization.

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The Annals of Statistics
2018, Vol. 46, No. 2, 642–678
https://doi.org/10.1214/17-AOS1563

GAUSSIAN AND BOOTSTRAP APPROXIMATIONS FOR HIGH-DIMENSIONAL U-STATISTICS AND THEIR APPLICATIONS[1]

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This paper studies the Gaussian and bootstrap approximations for the probabilities of a nondegenerate U-statistic belonging to the hyperrectangles in $\mathbb{R}^d$ when the dimension d is large. A two-step Gaussian approximation procedure that does not impose structural assumptions on the data distribution is proposed. Subject to mild moment conditions on the kernel, we establish the explicit rate of convergence uniformly in the class of all hyperrectangles in $\mathbb{R}^d$ that decays polynomially in sample size for a high-dimensional scaling limit, where the dimension can be much larger than the sample size. We also provide computable approximation methods for the quantiles of the maxima of centered U-statistics. Specifically, we provide a unified perspective for the empirical bootstrap, the randomly reweighted bootstrap and the Gaussian multiplier bootstrap with the jackknife estimator of covariance matrix as randomly reweighted quadratic forms and we establish their validity. We show that all three methods are inferentially first-order equivalent for high-dimensional U-statistics in the sense that they achieve the same uniform rate of convergence over all d-dimensional hyperrectangles. In particular, they are asymptotically valid when the dimension d can be as large as $O(e^{n^c})$ for some constant $c \in (0, 1/7)$.

The bootstrap methods are applied to statistical applications for high-dimensional non-Gaussian data including: (i) principled and data-dependent tuning parameter selection for regularized estimation of the covariance matrix and its related functionals; (ii) simultaneous inference for the covariance and rank correlation matrices. In particular, for the thresholded covariance matrix estimator with the bootstrap selected tuning parameter, we show that for a class of sub-Gaussian data, error bounds of the bootstrapped thresholded covariance matrix estimator can be much tighter than those of the minimax estimator with a universal threshold. In addition, we also show that the Gaussian-like convergence rates can be achieved for heavy-tailed data, which are less conservative than those obtained by the Bonferroni technique that ignores the dependency in the underlying data distribution.

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MSC2010 subject classifications. Primary 62E17; secondary 62F40.

Key words and phrases. U-statistics, high-dimensional inference, Gaussian approximation, bootstrap.

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The Annals of Statistics
2018, Vol. 46, No. 2, 679–710
https://doi.org/10.1214/17-AOS1564

SELECTIVE INFERENCE WITH A RANDOMIZED RESPONSE

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Inspired by sample splitting and the reusable holdout introduced in the field of differential privacy, we consider selective inference with a randomized response. We discuss two major advantages of using a randomized response for model selection. First, the selectively valid tests are more powerful after randomized selection. Second, it allows consistent estimation and weak convergence of selective inference procedures. Under independent sampling, we prove a selective (or privatized) central limit theorem that transfers procedures valid under asymptotic normality without selection to their corresponding selective counterparts. This allows selective inference in nonparametric settings. Finally, we propose a framework of inference after combining multiple randomized selection procedures. We focus on the classical asymptotic setting, leaving the interesting high-dimensional asymptotic questions for future work.

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MSC2010 subject classifications. Primary 62M40; secondary 62J05.
Key words and phrases. Selective inference, nonparametric, differential privacy.

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The Annals of Statistics
2018, Vol. 46, No. 2, 711–744
https://doi.org/10.1214/17-AOS1565

MULTISCALE BLIND SOURCE SEPARATION

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We provide a new methodology for statistical recovery of single linear mixtures of piecewise constant signals (sources) with unknown mixing weights and change points in a multiscale fashion. We show exact recovery within an ε-neighborhood of the mixture when the sources take only values in a known finite alphabet. Based on this we provide the SLAM (Separates Linear Alphabet Mixtures) estimators for the mixing weights and sources. For Gaussian error, we obtain uniform confidence sets and optimal rates (up to log-factors) for all quantities. SLAM is efficiently computed as a nonconvex optimization problem by a dynamic program tailored to the finite alphabet assumption. Its performance is investigated in a simulation study. Finally, it is applied to assign copy-number aberrations from genetic sequencing data to different clones and to estimate their proportions.

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MSC2010 subject classifications. Primary 62G08, 62G15; secondary 92D10.
Key words and phrases. Multiscale inference, honest confidence sets, change point regression, finite alphabet linear mixture, exact recovery, genetic sequencing.

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The Annals of Statistics
2018, Vol. 46, No. 2, 745–780
https://doi.org/10.1214/17-AOS1566

SHARP ORACLE INEQUALITIES FOR LEAST SQUARES ESTIMATORS IN SHAPE RESTRICTED REGRESSION[1]

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The performance of Least Squares (LS) estimators is studied in shape-constrained regression models under Gaussian and sub-Gaussian noise. General bounds on the performance of LS estimators over closed convex sets are provided. These results have the form of sharp oracle inequalities that account for the model misspecification error. In the presence of misspecification, these bounds imply that the LS estimator estimates the projection of the true parameter at the same rate as in the well-specified case.

In isotonic and unimodal regression, the LS estimator achieves the nonparametric rate $n^{-2/3}$ as well as a parametric rate of order k/n up to logarithmic factors, where k is the number of constant pieces of the true parameter. In univariate convex regression, the LS estimator satisfies an adaptive risk bound of order q/n up to logarithmic factors, where q is the number of affine pieces of the true regression function. This adaptive risk bound holds for any collection of design points. While Guntuboyina and Sen [*Probab. Theory Related Fields* **163** (2015) 379–411] established that the nonparametric rate of convex regression is of order $n^{-4/5}$ for equispaced design points, we show that the nonparametric rate of convex regression can be as slow as $n^{-2/3}$ for some worst-case design points. This phenomenon can be explained as follows: Although convexity brings more structure than unimodality, for some worst-case design points this extra structure is uninformative and the nonparametric rates of unimodal regression and convex regression are both $n^{-2/3}$. Higher order cones, such as the cone of β-monotone sequences, are also studied.

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MSC2010 subject classifications. 62G08, 62C20.
Key words and phrases. Shape restricted regression, convexity, minimax rates, Gaussian width, concentration.

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The Annals of Statistics
2018, Vol. 46, No. 2, 781–813
https://doi.org/10.1214/17-AOS1567

ORACLE INEQUALITIES FOR SPARSE ADDITIVE QUANTILE REGRESSION IN REPRODUCING KERNEL HILBERT SPACE

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This paper considers the estimation of the sparse additive quantile regression (SAQR) in high-dimensional settings. Given the nonsmooth nature of the quantile loss function and the nonparametric complexities of the component function estimation, it is challenging to analyze the theoretical properties of ultrahigh-dimensional SAQR. We propose a regularized learning approach with a two-fold Lasso-type regularization in a reproducing kernel Hilbert space (RKHS) for SAQR. We establish nonasymptotic oracle inequalities for the excess risk of the proposed estimator without any coherent conditions. If additional assumptions including an extension of the restricted eigenvalue condition are satisfied, the proposed method enjoys sharp oracle rates without the light tail requirement. In particular, the proposed estimator achieves the minimax lower bounds established for sparse additive mean regression. As a by-product, we also establish the concentration inequality for estimating the population mean when the general Lipschitz loss is involved. The practical effectiveness of the new method is demonstrated by competitive numerical results.

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MSC2010 subject classifications. Primary 62G20; secondary 62G05.

Key words and phrases. Quantile regression, additive models, sparsity, regularization methods, reproducing kernel Hilbert space.

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The Annals of Statistics
2018, Vol. 46, No. 2, 814–841
https://doi.org/10.1214/17-AOS1568

I-LAMM FOR SPARSE LEARNING: SIMULTANEOUS CONTROL OF ALGORITHMIC COMPLEXITY AND STATISTICAL ERROR[1]

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We propose a computational framework named iterative local adaptive majorize-minimization (I-LAMM) to simultaneously control algorithmic complexity and statistical error when fitting high-dimensional models. I-LAMM is a two-stage algorithmic implementation of the local linear approximation to a family of folded concave penalized quasi-likelihood. The first stage solves a convex program with a crude precision tolerance to obtain a coarse initial estimator, which is further refined in the second stage by iteratively solving a sequence of convex programs with smaller precision tolerances. Theoretically, we establish a phase transition: the first stage has a sublinear iteration complexity, while the second stage achieves an improved linear rate of convergence. Though this framework is completely algorithmic, it provides solutions with optimal statistical performances and controlled algorithmic complexity for a large family of nonconvex optimization problems. The iteration effects on statistical errors are clearly demonstrated via a contraction property. Our theory relies on a localized version of the sparse/restricted eigenvalue condition, which allows us to analyze a large family of loss and penalty functions and provide optimality guarantees under very weak assumptions (e.g., I-LAMM requires much weaker minimal signal strength than other procedures). Thorough numerical results are provided to support the obtained theory.

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MSC2010 subject classifications. Primary 62J07; secondary 62C20, 62H35.

Key words and phrases. Algorithmic statistics, iteration complexity, local adaptive MM, nonconvex statistical optimization, optimal rate of convergence.

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The Annals of Statistics
2018, Vol. 46, No. 2, 842–865
https://doi.org/10.1214/17-AOS1569

ON BAYESIAN INDEX POLICIES FOR SEQUENTIAL RESOURCE ALLOCATION

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This paper is about index policies for minimizing (frequentist) regret in a stochastic multi-armed bandit model, inspired by a Bayesian view on the problem. Our main contribution is to prove that the Bayes-UCB algorithm, which relies on quantiles of posterior distributions, is asymptotically optimal when the reward distributions belong to a one-dimensional exponential family, for a large class of prior distributions. We also show that the Bayesian literature gives new insight on what kind of exploration rates could be used in frequentist, UCB-type algorithms. Indeed, approximations of the Bayesian optimal solution or the Finite-Horizon Gittins indices provide a justification for the kl-UCB$^+$ and kl-UCB-H$^+$ algorithms, whose asymptotic optimality is also established.

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MSC2010 subject classifications. 62L05.
Key words and phrases. Multi-armed bandit problems, Bayesian methods, upper-confidence bounds, Gittins indices.

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The Annals of Statistics
2018, Vol. 46, No. 2, 866–894
https://doi.org/10.1214/17-AOS1571

TESTING INDEPENDENCE WITH HIGH-DIMENSIONAL CORRELATED SAMPLES

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Testing independence among a number of (ultra) high-dimensional random samples is a fundamental and challenging problem. By arranging n identically distributed p-dimensional random vectors into a $p \times n$ data matrix, we investigate the problem of testing independence among columns under the matrix-variate normal modeling of data. We propose a computationally simple and tuning-free test statistic, characterize its limiting null distribution, analyze the statistical power and prove its minimax optimality. As an important by-product of the test statistic, a ratio-consistent estimator for the quadratic functional of a covariance matrix from correlated samples is developed. We further study the effect of correlation among samples to an important high-dimensional inference problem—large-scale multiple testing of Pearson's correlation coefficients. Indeed, blindly using classical inference results based on the assumed independence of samples will lead to many false discoveries, which suggests the need for conducting independence testing before applying existing methods. To address the challenge arising from correlation among samples, we propose a "sandwich estimator" of Pearson's correlation coefficient by de-correlating the samples. Based on this approach, the resulting multiple testing procedure asymptotically controls the overall false discovery rate at the nominal level while maintaining good statistical power. Both simulated and real data experiments are carried out to demonstrate the advantages of the proposed methods.

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MSC2010 subject classifications. Primary 62F05; secondary 62H10.

Key words and phrases. Independence test, multiple testing of correlations, false discovery rate, matrix-variate normal, quadratic functional estimation, high-dimensional sample correlation matrix.

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The Annals of Statistics
2018, Vol. 46, No. 2, 895–923
https://doi.org/10.1214/17-AOS1574

DETECTING RARE AND FAINT SIGNALS VIA THRESHOLDING MAXIMUM LIKELIHOOD ESTIMATORS[1]

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Motivated by the analysis of RNA sequencing (RNA-seq) data for genes differentially expressed across multiple conditions, we consider detecting rare and faint signals in high-dimensional response variables. We address the signal detection problem under a general framework, which includes generalized linear models for count-valued responses as special cases. We propose a test statistic that carries out a multi-level thresholding on maximum likelihood estimators (MLEs) of the signals, based on a new Cramér-type moderate deviation result for multidimensional MLEs. Based on the multi-level thresholding test, a multiple testing procedure is proposed for signal identification. Numerical simulations and a case study on maize RNA-seq data are conducted to demonstrate the effectiveness of the proposed approaches on signal detection and identification.

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MSC2010 subject classifications. Primary 62H15; secondary 62G20, 62G32.

Key words and phrases. Detection boundary, false discovery proportion, generalized linear model, moderate deviation, multiple testing procedure, RNA-seq data.

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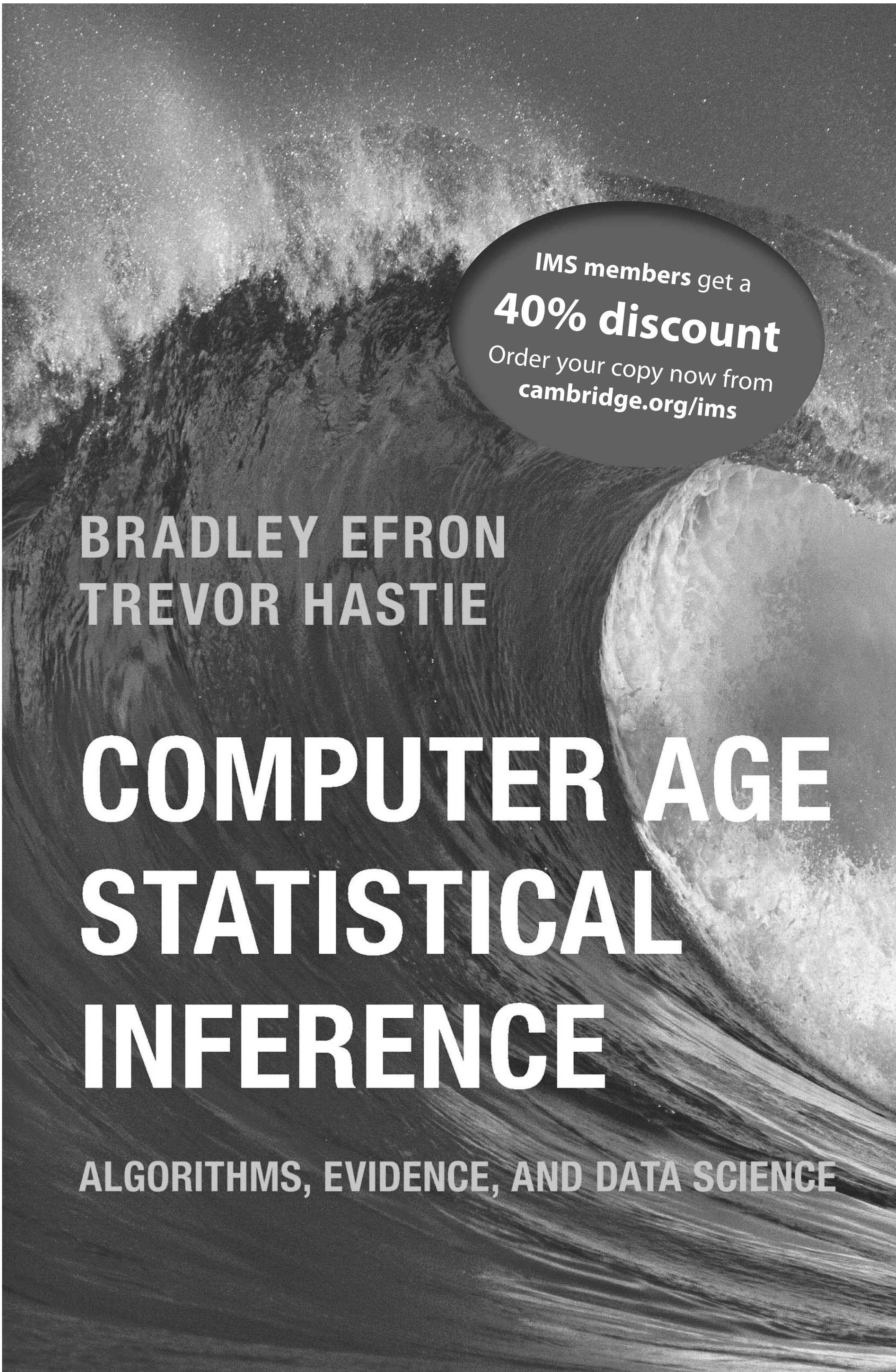
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