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ON THE OPTIMALITY OF SLICED INVERSE REGRESSION IN HIGH DIMENSIONS

BY QIAN LIN[1], XINRAN LI[2], DONGMING HUANG[3] AND JUN S. LIU[4]

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The central subspace of a pair of random variables $(y, \boldsymbol{x}) \in \mathbb{R}^{p+1}$ is the minimal subspace $\mathcal{S}$ such that $y \perp\!\!\!\perp \boldsymbol{x} | P_{\mathcal{S}}\boldsymbol{x}$. In this paper, we consider the minimax rate of estimating the central space under the multiple index model $y = f(\boldsymbol{\beta}_1^\tau \boldsymbol{x}, \boldsymbol{\beta}_2^\tau \boldsymbol{x}, \dots, \boldsymbol{\beta}_d^\tau \boldsymbol{x}, \epsilon)$ with at most s active predictors, where $\boldsymbol{x} \sim N(0, \boldsymbol{\Sigma})$ for some class of $\boldsymbol{\Sigma}$. We first introduce a large class of models depending on the smallest nonzero eigenvalue λ of $\mathrm{var}(\mathbb{E}[\boldsymbol{x}|y])$, over which we show that an aggregated estimator based on the SIR procedure converges at rate $d \wedge ((sd + s\log(ep/s))/(n\lambda))$. We then show that this rate is optimal in two scenarios, the single index models and the multiple index models with fixed central dimension d and fixed λ. By assuming a technical conjecture, we can show that this rate is also optimal for multiple index models with bounded dimension of the central space.

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ASYMPTOTIC OPTIMALITY IN STOCHASTIC OPTIMIZATION

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We study local complexity measures for stochastic convex optimization problems, providing a local minimax theory analogous to that of Hájek and Le Cam for classical statistical problems. We give complementary optimality results, developing fully online methods that adaptively achieve optimal convergence guarantees. Our results provide function-specific lower bounds and convergence results that make precise a correspondence between statistical difficulty and the geometric notion of tilt-stability from optimization. As part of this development, we show how variants of Nesterov's dual averaging—a stochastic gradient-based procedure—guarantee finite time identification of constraints in optimization problems, while stochastic gradient procedures fail. Additionally, we highlight a gap between problems with linear and nonlinear constraints: standard stochastic-gradient-based procedures are suboptimal even for the simplest nonlinear constraints, necessitating the development of asymptotically optimal Riemannian stochastic gradient methods.

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CONCORDANCE AND VALUE INFORMATION CRITERIA FOR OPTIMAL TREATMENT DECISION

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Personalized medicine is a medical procedure that receives considerable scientific and commercial attention. The goal of personalized medicine is to assign the optimal treatment regime for each individual patient, according to his/her personal prognostic information. When there are a large number of pretreatment variables, it is crucial to identify those important variables that are necessary for treatment decision making. In this paper, we study two information criteria: the concordance and value information criteria, for variable selection in optimal treatment decision making. We consider both fixed-p and high dimensional settings, and show our information criteria are consistent in model/tuning parameter selection. We further apply our information criteria to four estimation approaches, including robust learning, concordance-assisted learning, penalized A-learning and sparse concordance-assisted learning, and demonstrate the empirical performance of our methods by simulations.

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STATISTICALLY OPTIMAL AND COMPUTATIONALLY EFFICIENT LOW RANK TENSOR COMPLETION FROM NOISY ENTRIES

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In this article, we develop methods for estimating a low rank tensor from noisy observations on a subset of its entries to achieve both statistical and computational efficiencies. There have been a lot of recent interests in this problem of noisy tensor completion. Much of the attention has been focused on the fundamental computational challenges often associated with problems involving higher order tensors, yet very little is known about their statistical performance. To fill in this void, in this article, we characterize the fundamental statistical limits of noisy tensor completion by establishing minimax optimal rates of convergence for estimating a kth order low rank tensor under the general ℓ_p $(1 \le p \le 2)$ norm which suggest significant room for improvement over the existing approaches. Furthermore, we propose a polynomial-time computable estimating procedure based upon power iteration and a second-order spectral initialization that achieves the optimal rates of convergence. Our method is fairly easy to implement and numerical experiments are presented to further demonstrate the practical merits of our estimator.

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TRANSFER LEARNING FOR NONPARAMETRIC CLASSIFICATION: MINIMAX RATE AND ADAPTIVE CLASSIFIER

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Human learners have the natural ability to use knowledge gained in one setting for learning in a different but related setting. This ability to transfer knowledge from one task to another is essential for effective learning. In this paper, we study transfer learning in the context of nonparametric classification based on observations from different distributions under the posterior drift model, which is a general framework and arises in many practical problems.

We first establish the minimax rate of convergence and construct a rate-optimal two-sample weighted K-NN classifier. The results characterize precisely the contribution of the observations from the source distribution to the classification task under the target distribution. A data-driven adaptive classifier is then proposed and is shown to simultaneously attain within a logarithmic factor of the optimal rate over a large collection of parameter spaces. Simulation studies and real data applications are carried out where the numerical results further illustrate the theoretical analysis. Extensions to the case of multiple source distributions are also considered.

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ADAPTATION IN MULTIVARIATE LOG-CONCAVE DENSITY ESTIMATION

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We study the adaptation properties of the multivariate log-concave maximum likelihood estimator over three subclasses of log-concave densities. The first consists of densities with polyhedral support whose logarithms are piecewise affine. The complexity of such densities f can be measured in terms of the sum $\Gamma(f)$ of the numbers of facets of the subdomains in the polyhedral subdivision of the support induced by f. Given n independent observations from a d-dimensional log-concave density with $d \in \{2, 3\}$, we prove a sharp oracle inequality, which in particular implies that the Kullback–Leibler risk of the log-concave maximum likelihood estimator for such densities is bounded above by $\Gamma(f)/n$, up to a polylogarithmic factor. Thus, the rate can be essentially parametric, even in this multivariate setting. For the second type of adaptation, we consider densities that are bounded away from zero on a polytopal support; we show that up to polylogarithmic factors, the log-concave maximum likelihood estimator attains the rate $n^{-4/7}$ when $d = 3$, which is faster than the worst-case rate of $n^{-1/2}$. Finally, our third type of subclass consists of densities whose contours are well separated; these new classes are constructed to be affine invariant and turn out to contain a wide variety of densities, including those that satisfy Hölder regularity conditions. Here, we prove another sharp oracle inequality, which reveals in particular that the log-concave maximum likelihood estimator attains a risk bound of order $n^{-\min(\frac{\beta+3}{\beta+7}, \frac{4}{7})}$ when $d = 3$ over the class of β-Hölder log-concave densities with $\beta > 1$, again up to a polylogarithmic factor.

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ASYMPTOTICALLY INDEPENDENT U-STATISTICS IN HIGH-DIMENSIONAL TESTING

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Many high-dimensional hypothesis tests aim to globally examine marginal or low-dimensional features of a high-dimensional joint distribution, such as testing of mean vectors, covariance matrices and regression coefficients. This paper constructs a family of U-statistics as unbiased estimators of the ℓ_p-norms of those features. We show that under the null hypothesis, the U-statistics of different finite orders are asymptotically independent and normally distributed. Moreover, they are also asymptotically independent with the maximum-type test statistic, whose limiting distribution is an extreme value distribution. Based on the asymptotic independence property, we propose an adaptive testing procedure which combines p-values computed from the U-statistics of different orders. We further establish power analysis results and show that the proposed adaptive procedure maintains high power against various alternatives.

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FREQUENTIST VALIDITY OF BAYESIAN LIMITS

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To the frequentist who computes posteriors, not all priors are useful asymptotically: in this paper, a Bayesian perspective on test sequences is proposed and Schwartz's Kullback–Leibler condition is generalised to widen the range of frequentist applications of posterior convergence. With *Bayesian tests* and a weakened form of contiguity termed *remote contiguity*, we prove simple and fully general frequentist theorems, for posterior consistency and rates of convergence, for consistency of posterior odds in model selection, and for conversion of sequences of credible sets into sequences of confidence sets with asymptotic coverage one. For frequentist uncertainty quantification, this means that a prior inducing remote contiguity allows one to enlarge credible sets of calculated, simulated or approximated posteriors to obtain asymptotically consistent confidence sets.

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OPTIMAL CHANGE POINT DETECTION AND LOCALIZATION IN SPARSE DYNAMIC NETWORKS

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We study the problem of change point localization in dynamic networks models. We assume that we observe a sequence of independent adjacency matrices of the same size, each corresponding to a realization of an unknown inhomogeneous Bernoulli model. The underlying distribution of the adjacency matrices are piecewise constant, and may change over a subset of the time points, called change points. We are concerned with recovering the unknown number and positions of the change points. In our model setting, we allow for all the model parameters to change with the total number of time points, including the network size, the minimal spacing between consecutive change points, the magnitude of the smallest change and the degree of sparsity of the networks. We first identify a region of impossibility in the space of the model parameters such that no change point estimator is provably consistent if the data are generated according to parameters falling in that region. We propose a computationally-simple algorithm for network change point localization, called network binary segmentation, that relies on weighted averages of the adjacency matrices. We show that network binary segmentation is consistent over a range of the model parameters that nearly cover the complement of the impossibility region, thus demonstrating the existence of a phase transition for the problem at hand. Next, we devise a more sophisticated algorithm based on singular value thresholding, called local refinement, that delivers more accurate estimates of the change point locations. Under appropriate conditions, local refinement guarantees a minimax optimal rate for network change point localization while remaining computationally feasible.

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CONVERGENCE OF COVARIANCE AND SPECTRAL DENSITY ESTIMATES FOR HIGH-DIMENSIONAL LOCALLY STATIONARY PROCESSES

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Covariances and spectral density functions play a fundamental role in the theory of time series. There is a well-developed asymptotic theory for their estimates for low-dimensional stationary processes. For high-dimensional non-stationary processes, however, many important problems on their asymptotic behaviors are still unanswered. This paper presents a systematic asymptotic theory for the estimates of time-varying second-order statistics for a general class of high-dimensional locally stationary processes. Using the framework of functional dependence measure, we derive convergence rates of the estimates which depend on the sample size T, the dimension p, the moment condition and the dependence of the underlying processes.

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WORDLENGTH ENUMERATOR FOR FRACTIONAL FACTORIAL DESIGNS

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While the minimum aberration criterion is popular for selecting good designs with qualitative factors under an ANOVA model, the minimum β-aberration criterion is more suitable for selecting designs with quantitative factors under a polynomial model. In this paper, we propose the concept of wordlength enumerator to unify these two criteria. The wordlength enumerator is defined as an average similarity of contrasts among all possible pairs of runs. The wordlength enumerator is easy and fast to compute, and can be used to compare and rank designs efficiently. Based on the wordlength enumerator, we develop simple and fast methods for calculating both the generalized wordlength pattern and the β-wordlength pattern. We further obtain a lower bound of the wordlength enumerator for three-level designs and characterize the combinatorial structure of designs achieving the lower bound. Finally, we propose two methods for constructing supersaturated designs that have both generalized minimum aberration and minimum β-aberration.

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ESTIMATING MINIMUM EFFECT WITH OUTLIER SELECTION

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We introduce one-sided versions of Huber's contamination model, in which corrupted samples tend to take larger values than uncorrupted ones. Two intertwined problems are addressed: estimation of the mean of the uncorrupted samples (minimum effect) and selection of the corrupted samples (outliers). Regarding estimation of the minimum effect, we derive the minimax risks and introduce estimators that are adaptive with respect to the unknown number of contaminations. The optimal convergence rates differ from the ones in the classical Huber contamination model. This fact uncovers the effect of the one-sided structural assumption of the contaminations. As for the problem of selecting the outliers, we formulate the problem in a multiple testing framework for which the location and scaling of the null hypotheses are unknown. We rigorously prove that estimating the null hypothesis while maintaining a theoretical guarantee on the amount of the falsely selected outliers is possible, both through false discovery rate (FDR) and through post hoc bounds. As a by-product, we address a long-standing open issue on FDR control under equi-correlation, which reinforces the interest of removing dependency in such a setting.

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MULTIPLE BLOCK SIZES AND OVERLAPPING BLOCKS FOR MULTIVARIATE TIME SERIES EXTREMES

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Block maxima methods constitute a fundamental part of the statistical toolbox in extreme value analysis. However, most of the corresponding theory is derived under the simplifying assumption that block maxima are independent observations from a genuine extreme value distribution. In practice, however, block sizes are finite and observations from different blocks are dependent. Theory respecting the latter complications is not well developed, and, in the multivariate case, has only recently been established for disjoint blocks of a single block size. We show that using overlapping blocks instead of disjoint blocks leads to a uniform improvement in the asymptotic variance of the multivariate empirical distribution function of rescaled block maxima and any smooth functionals thereof (such as the empirical copula), without any sacrifice in the asymptotic bias. We further derive functional central limit theorems for multivariate empirical distribution functions and empirical copulas that are uniform in the block size parameter, which seems to be the first result of this kind for estimators based on block maxima in general. The theory allows for various aggregation schemes over multiple block sizes, leading to substantial improvements over the single block length case and opens the door to further methodology developments. In particular, we consider bias correction procedures that can improve the convergence rates of extreme-value estimators and shed some new light on estimation of the second-order parameter when the main purpose is bias correction.

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ESTIMATION OF LOW-RANK MATRICES VIA APPROXIMATE MESSAGE PASSING

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Consider the problem of estimating a low-rank matrix when its entries are perturbed by Gaussian noise, a setting that is also known as "spiked model" or "deformed random matrix." If the empirical distribution of the entries of the spikes is known, optimal estimators that exploit this knowledge can substantially outperform simple spectral approaches. Recent work characterizes the asymptotic accuracy of Bayes-optimal estimators in the high-dimensional limit. In this paper, we present a practical algorithm that can achieve Bayes-optimal accuracy above the spectral threshold. A bold conjecture from statistical physics posits that no polynomial-time algorithm achieves optimal error below the same threshold (unless the best estimator is trivial).

Our approach uses Approximate Message Passing (AMP) in conjunction with a spectral initialization. AMP algorithms have proved successful in a variety of statistical estimation tasks, and are amenable to exact asymptotic analysis via state evolution. Unfortunately, state evolution is uninformative when the algorithm is initialized near an unstable fixed point, as often happens in low-rank matrix estimation problems. We develop a new analysis of AMP that allows for spectral initializations, and builds on a decoupling between the outlier eigenvectors and the bulk in the spiked random matrix model.

Our main theorem is general and applies beyond matrix estimation. However, we use it to derive detailed predictions for the problem of estimating a rank-one matrix in noise. Special cases of this problem are closely related—via universality arguments—to the network community detection problem for two asymmetric communities. For general rank-one models, we show that AMP can be used to construct confidence intervals and control false discovery rate.

We provide illustrations of the general methodology by considering the cases of sparse low-rank matrices and of block-constant low-rank matrices with symmetric blocks (we refer to the latter as to the "Gaussian block model").

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ASYMPTOTICS FOR SPHERICAL FUNCTIONAL AUTOREGRESSIONS

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In this paper, we investigate a class of spherical functional autoregressive processes, and we discuss the estimation of the corresponding autoregressive kernels. In particular, we first establish a consistency result (in mean-square and sup norm), then a quantitative central limit theorem (in Wasserstein distance), and finally a weak convergence result, under more restrictive regularity conditions. Our results are validated by a small numerical investigation.

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SINGULAR VECTOR AND SINGULAR SUBSPACE DISTRIBUTION FOR THE MATRIX DENOISING MODEL

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In this paper, we study the matrix denoising model $Y = S + X$, where S is a low rank deterministic signal matrix and X is a random noise matrix, and both are $M \times n$. In the scenario that M and n are comparably large and the signals are supercritical, we study the fluctuation of the outlier singular vectors of Y, under fully general assumptions on the structure of S and the distribution of X. More specifically, we derive the limiting distribution of angles between the principal singular vectors of Y and their deterministic counterparts, the singular vectors of S. Further, we also derive the distribution of the distance between the subspace spanned by the principal singular vectors of Y and that spanned by the singular vectors of S. It turns out that the limiting distributions depend on the structure of the singular vectors of S and the distribution of X, and thus they are nonuniversal. Statistical applications of our results to singular vector and singular subspace inferences are also discussed.

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ROBUST MULTIVARIATE MEAN ESTIMATION: THE OPTIMALITY OF TRIMMED MEAN

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We consider the problem of estimating the mean of a random vector based on i.i.d. observations and adversarial contamination. We introduce a multivariate extension of the trimmed-mean estimator and show its optimal performance under minimal conditions.

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CLASSIFICATION ACCURACY AS A PROXY FOR TWO-SAMPLE TESTING

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When data analysts train a classifier and check if its accuracy is significantly different from chance, they are implicitly performing a two-sample test. We investigate the statistical properties of this flexible approach in the high-dimensional setting. We prove two results that hold for all classifiers in any dimensions: if its true error remains ϵ-better than chance for some $\epsilon > 0$ as $d, n \to \infty$, then (a) the permutation-based test is consistent (has power approaching to one), (b) a computationally efficient test based on a Gaussian approximation of the null distribution is also consistent. To get a finer understanding of the rates of consistency, we study a specialized setting of distinguishing Gaussians with mean-difference δ and common (known or unknown) covariance Σ, when $d/n \to c \in (0, \infty)$. We study variants of Fisher's linear discriminant analysis (LDA) such as "naive Bayes" in a nontrivial regime when $\epsilon \to 0$ (the Bayes classifier has true accuracy approaching 1/2), and contrast their power with corresponding variants of Hotelling's test. Surprisingly, the expressions for their power match exactly in terms of n, d, δ, Σ, and the LDA approach is only worse by a constant factor, achieving an asymptotic relative efficiency (ARE) of $1/\sqrt{\pi}$ for balanced samples. We also extend our results to high-dimensional elliptical distributions with finite kurtosis. Other results of independent interest include minimax lower bounds, and the optimality of Hotelling's test when $d = o(n)$. Simulation results validate our theory, and we present practical takeaway messages along with natural open problems.

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ASYMMETRY HELPS: EIGENVALUE AND EIGENVECTOR ANALYSES OF ASYMMETRICALLY PERTURBED LOW-RANK MATRICES[1]

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This paper is concerned with the interplay between statistical asymmetry and spectral methods. Suppose we are interested in estimating a rank-1 and symmetric matrix $\boldsymbol{M}^\star \in \mathbb{R}^{n\times n}$, yet only a randomly perturbed version $\boldsymbol{M}$ is observed. The noise matrix $\boldsymbol{M} - \boldsymbol{M}^\star$ is composed of independent (but not necessarily homoscedastic) entries and is, therefore, not symmetric in general. This might arise if, for example, when we have two independent samples for each entry of $\boldsymbol{M}^\star$ and arrange them in an *asymmetric* fashion. The aim is to estimate the leading eigenvalue and the leading eigenvector of $\boldsymbol{M}^\star$.

We demonstrate that the leading eigenvalue of the data matrix $\boldsymbol{M}$ can be $O(\sqrt{n})$ times more accurate (up to some log factor) than its (unadjusted) leading singular value of $\boldsymbol{M}$ in eigenvalue estimation. Moreover, the eigendecomposition approach is fully adaptive to heteroscedasticity of noise, without the need of any prior knowledge about the noise distributions. In a nutshell, this curious phenomenon arises since the statistical asymmetry automatically mitigates the bias of the eigenvalue approach, thus eliminating the need of careful bias correction. Additionally, we develop appealing nonasymptotic eigenvector perturbation bounds; in particular, we are able to bound the perturbation of any linear function of the leading eigenvector of $\boldsymbol{M}$ (e.g., entrywise eigenvector perturbation). We also provide partial theory for the more general rank-r case. The takeaway message is this: arranging the data samples in an asymmetric manner and performing eigendecomposition could sometimes be quite beneficial.

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COMPLEX SAMPLING DESIGNS: UNIFORM LIMIT THEOREMS AND APPLICATIONS

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In this paper, we develop a general approach to proving global and local uniform limit theorems for the Horvitz–Thompson empirical process arising from complex sampling designs. Global theorems such as Glivenko–Cantelli and Donsker theorems, and local theorems such as local asymptotic modulus and related ratio-type limit theorems are proved for both the Horvitz–Thompson empirical process, and its calibrated version. Limit theorems of other variants and their conditional versions are also established. Our approach reveals an interesting feature: the problem of deriving uniform limit theorems for the Horvitz–Thompson empirical process is essentially no harder than the problem of establishing the corresponding finite-dimensional limit theorems, once the usual complexity conditions on the function class are satisfied. These global and local uniform limit theorems are then applied to important statistical problems including (i) M-estimation, (ii) Z-estimation and (iii) frequentist theory of pseudo-Bayes procedures, all with weighted likelihood, to illustrate their wide applicability.

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PREDICTIVE INFERENCE WITH THE JACKKNIFE+

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This paper introduces the *jackknife+*, which is a novel method for constructing predictive confidence intervals. Whereas the jackknife outputs an interval centered at the predicted response of a test point, with the width of the interval determined by the quantiles of leave-one-out residuals, the jackknife+ also uses the leave-one-out predictions at the test point to account for the variability in the fitted regression function. Assuming exchangeable training samples, we prove that this crucial modification permits rigorous coverage guarantees regardless of the distribution of the data points, for any algorithm that treats the training points symmetrically. Such guarantees are not possible for the original jackknife and we demonstrate examples where the coverage rate may actually vanish. Our theoretical and empirical analysis reveals that the jackknife and the jackknife+ intervals achieve nearly exact coverage and have similar lengths whenever the fitting algorithm obeys some form of stability. Further, we extend the jackknife+ to K-fold cross validation and similarly establish rigorous coverage properties. Our methods are related to *cross-conformal prediction* proposed by Vovk (*Ann. Math. Artif. Intell.* **74** (2015) 9–28) and we discuss connections.

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ROBUST ESTIMATION OF SUPERHEDGING PRICES

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We consider statistical estimation of superhedging prices using historical stock returns in a frictionless market with d traded assets. We introduce a plug-in estimator based on empirical measures and show it is consistent but lacks suitable robustness. To address this, we propose novel estimators which use a larger set of martingale measures defined through a tradeoff between the radius of Wasserstein balls around the empirical measure and the allowed norm of martingale densities. We then extend our study, in part, to estimation of risk measures, to the case of markets with traded options, to a multi-period setting and to settings with model uncertainty. We also study convergence rates of estimators and convergence of super-hedging strategies.

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CONCENTRATION OF KERNEL MATRICES WITH APPLICATION TO KERNEL SPECTRAL CLUSTERING

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We study the concentration of random kernel matrices around their mean. We derive nonasymptotic exponential concentration inequalities for Lipschitz kernels assuming that the data points are independent draws from a class of multivariate distributions on $\mathbb{R}^d$, including the strongly log-concave distributions under affine transformations. A feature of our result is that the data points need not have identical distributions or zero mean, which is key in certain applications such as clustering. Our bound for the Lipschitz kernels is dimension-free and sharp up to constants. For comparison, we also derive the companion result for the Euclidean (inner product) kernel for a class of sub-Gaussian distributions. A notable difference between the two cases is that, in contrast to the Euclidean kernel, in the Lipschitz case, the concentration inequality does not depend on the mean of the underlying vectors. As an application of these inequalities, we derive a bound on the misclassification rate of a kernel spectral clustering (KSC) algorithm, under a perturbed nonparametric mixture model. We show an example where this bound establishes the high-dimensional consistency (as $d \to \infty$) of the KSC, when applied with a Gaussian kernel, to a noisy model of nested nonlinear manifolds.

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A RULE OF THUMB: RUN LENGTHS TO FALSE ALARM OF MANY TYPES OF CONTROL CHARTS RUN IN PARALLEL ON DEPENDENT STREAMS ARE ASYMPTOTICALLY INDEPENDENT

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Consider a process that produces a series of independent identically distributed vectors. A change in an underlying state may become manifest in a modification of one or more of the marginal distributions. Often, the dependence structure between coordinates is unknown, impeding surveillance based on the joint distribution. A popular approach is to construct control charts for each coordinate separately and raise an alarm the first time any (or some) of the control charts signals. The difficulty is obtaining an expression for the overall average run length to false alarm (ARL2FA).

We argue that despite the dependence structure, when the process is in control, for large ARLs to false alarm, run lengths of many types of control charts run in parallel are asymptotically independent. Furthermore, often, in-control run lengths are asymptotically exponentially distributed, enabling uncomplicated asymptotic expressions for the ARL2FA.

We prove this assertion for certain Cusum and Shiryaev–Roberts-type control charts and illustrate it by simulations.

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SHARP MINIMAX DISTRIBUTION ESTIMATION FOR CURRENT STATUS CENSORING WITH OR WITHOUT MISSING

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Nonparametric estimation of the cumulative distribution function and the probability density of a lifetime X modified by a current status censoring (CSC), including cases of right and left missing data, is a classical ill-posed problem with biased data. The biased nature of CSC data may preclude us from consistent estimation unless the biasing function is known or may be estimated, and its ill-posed nature slows down rates of convergence. Under a traditionally studied CSC, we observe a sample from (Z, Δ) where a continuous monitoring time Z is independent of X, $\Delta := I(X \le Z)$ is the status, and the bias of observations is created by the density of Z which is estimable. In presence of right or left missing, we observe corresponding samples from $(\Delta Z, \Delta)$ or $((1 - \Delta)Z, \Delta)$; the data are again biased but now the density of Z cannot be estimated from the data. As a result, to solve the estimation problem, either the density of Z must be known (like in a controlled study) or an extra cross-sectional sampling of Z, which is typically simpler than an underlying CSC study, be conducted. The main aim of the paper is to develop for this biased and ill-posed problem the theory of efficient (sharp-minimax) estimation which is inspired by known results for the case of directly observed X. Among interesting aspects of the developed theory: (i) While sharp-minimax analysis of missing CSC may follow the classical Pinsker's methodology, analysis of CSC requires a more complicated estimation procedure based on a special smoothing in both frequency and time domains; (ii) Efficient estimation requires solving an old-standing problem of approximating aperiodic Sobolev functions; (iii) If smoothness of the cdf of X is known, then its rate-minimax estimation is possible even if the density of Z is rougher. Real and simulated examples, as well as extensions of the core models to dependent X and Z and case-control CSC, are presented.

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WASSERSTEIN F-TESTS AND CONFIDENCE BANDS FOR THE FRÉCHET REGRESSION OF DENSITY RESPONSE CURVES

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Data consisting of samples of probability density functions are increasingly prevalent, necessitating the development of methodologies for their analysis that respect the inherent nonlinearities associated with densities. In many applications, density curves appear as functional response objects in a regression model with vector predictors. For such models, inference is key to understand the importance of density-predictor relationships, and the uncertainty associated with the estimated conditional mean densities, defined as conditional Fréchet means under a suitable metric. Using the Wasserstein geometry of optimal transport, we consider the Fréchet regression of density curve responses and develop tests for global and partial effects, as well as simultaneous confidence bands for estimated conditional mean densities. The asymptotic behavior of these objects is based on underlying functional central limit theorems within Wasserstein space, and we demonstrate that they are asymptotically of the correct size and coverage, with uniformly strong consistency of the proposed tests under sequences of contiguous alternatives. The accuracy of these methods, including nominal size, power and coverage, is assessed through simulations, and their utility is illustrated through a regression analysis of post-intracerebral hemorrhage hematoma densities and their associations with a set of clinical and radiological covariates.

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CORRECTION NOTE: "OPTIMAL TWO-STAGE PROCEDURES FOR ESTIMATING LOCATION AND SIZE OF THE MAXIMUM OF A MULTIVARIATE REGRESSION FUNCTION" *Ann. Statist.* 40 (2012) 2850–2876

BY EDUARD BELITSER[1,*] SUBHASHIS GHOSAL[2] AND HARRY VAN ZANTEN[1,†]

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We rectify a wrongly stated fact in the paper of Belitser, Ghosal and van Zanten (*Ann. Statist.* **40** (2012) 2850–2876).

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CORRECTION NOTE: HIGHER ORDER ELICITABILITY AND OSBAND'S PRINCIPLE

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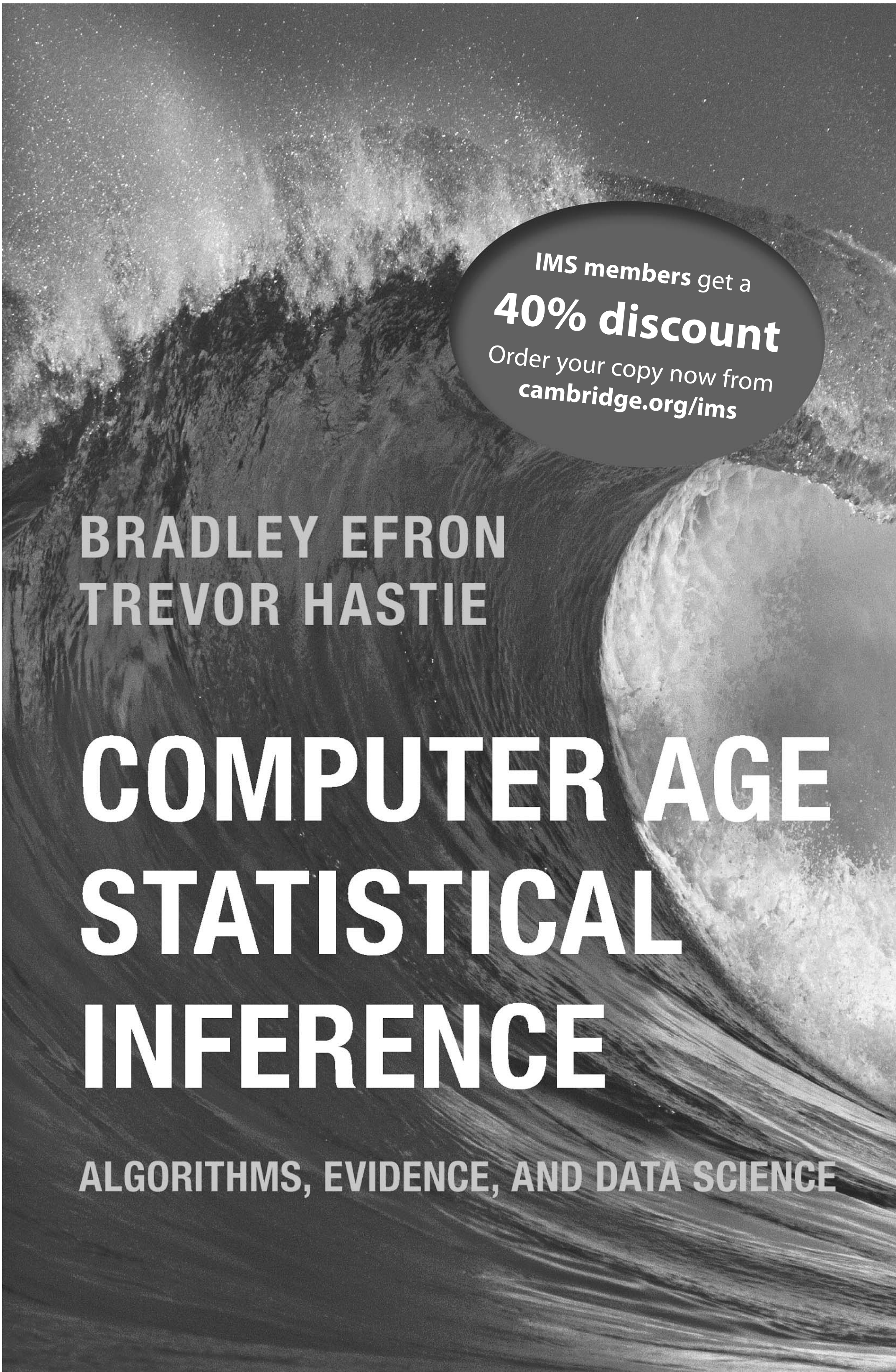
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