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ASYMPTOTIC DISTRIBUTIONS OF LARGEST PEARSON CORRELATION COEFFICIENTS UNDER DEPENDENT STRUCTURES

BY TIEFENG JIANG[1,a] AND TUAN PHAM[2,b]

[1]*School of Data Science, Chinese University of Hong Kong,* [a]*jiang040@cuhk.edu.cn*
[2]*Department of Statistics and Data Science, University of Texas,* [b]*tuan.pham@utexas.edu*

Given a random sample from a multivariate normal distribution whose covariance matrix is a Toeplitz matrix, we study the largest off-diagonal entry of the sample correlation matrix. Assuming the multivariate normal distribution has the covariance structure of an autoregressive sequence, we establish a phase transition in the limiting distribution of the largest off-diagonal entry. We show that the limiting distributions are of Gumbel-type (with different parameters), depending on how large or small the parameter of the autoregressive sequence is. At the critical case, we obtain that the limiting distribution is the maximum of two independent random variables of Gumbel distributions. This phase transition establishes the exact threshold at which the autoregressive covariance structure behaves differently than its counterpart with the covariance matrix equal to the identity. Assuming the covariance matrix is a general Toeplitz matrix, we obtain the limiting distribution of the largest entry under the ultrahigh-dimensional settings: it is a weighted sum of two independent random variables, one normal and the other following a Gumbel-type law. The counterpart of the non-Gaussian case is also discussed. We use our results to obtain new testing procedures for some problems in high-dimensional statistics.

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ON THE CONVERGENCE OF COORDINATE ASCENT VARIATIONAL INFERENCE

BY ANIRBAN BHATTACHARYA[1,a], DEBDEEP PATI[2,b] AND YUN YANG[3,c]

[1]*Department of Statistics, Texas A&M University,* [a]*anirbanb@stat.tamu.edu*
[2]*Department of Statistics, University of Wisconsin-Madison,* [b]*dpati2@wisc.edu*
[3]*Department of Mathematics, University of Maryland at College Park,* [c]*yy84@umd.edu*

As a computational alternative to Markov chain Monte Carlo approaches, variational inference (VI) is becoming more and more popular for approximating intractable posterior distributions in large-scale Bayesian models due to its comparable efficacy and superior efficiency. Several recent works provide theoretical justifications of VI by proving its statistical optimality for parameter estimation under various settings; meanwhile, formal analysis on the algorithmic convergence aspects of VI is still largely lacking. In this paper, we consider the common coordinate ascent variational inference (CAVI) algorithm for implementing the mean-field (MF) VI towards optimizing a Kullback–Leibler divergence objective functional over the space of all factorized distributions. Focusing on the two-block case, we analyze the convergence of CAVI by leveraging the extensive toolbox from functional analysis and optimization. We provide general conditions for certifying global or local exponential convergence of CAVI. Specifically, a new notion of generalized correlation for characterizing the interaction between the constituting blocks in influencing the VI objective functional is introduced, which according to the theory, quantifies the algorithmic contraction rate of two-block CAVI. As illustrations, we apply the theory to a number of examples, and derive explicit problem-dependent upper bounds on the algorithmic contraction rate.

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OPTIMAL TRANSPORT MAP ESTIMATION IN GENERAL FUNCTION SPACES

BY VINCENT DIVOL[1,a], JONATHAN NILES-WEED[2,3,b] AND ARAM-ALEXANDRE POOLADIAN[3,c]

[1]*CREST, ENSAE, IP Paris,* [a]*vincent.divol@ensae.fr*
[2]*Courant Institute of Mathematical Sciences, New York University,* [b]*jnw@cims.nyu.edu,* [c]*aram-alexandre.pooladian@nyu.edu*
[3]*Center for Data Science, New York University*

We study the problem of estimating a function T, given independent samples from a distribution P and from the pushforward distribution $T_\sharp P$. This setting is motivated by applications in the sciences, where T represents the evolution of a physical system over time, and in machine learning, where, for example, T may represent a transformation learned by a deep neural network trained for a generative modeling task. To ensure identifiability, we assume that $T = \nabla\varphi_0$ is the gradient of a convex function in which case T is known as an *optimal transport map*. Prior work has studied the estimation of T under the assumption that it lies in a Hölder class, but general theory is lacking. We present a unified methodology for obtaining rates of estimation of optimal transport maps in general function spaces. Our assumptions are significantly weaker than those appearing in the literature: we require only that the source measure P satisfy a Poincaré inequality and that the optimal map be the gradient of a smooth convex function that lies in a space whose metric entropy can be controlled. As a special case, we recover known estimation rates for Hölder transport maps but also obtain nearly sharp results in many settings not covered by prior work. For example, we provide the first statistical rates of estimation when P is the normal distribution and the transport map is given by an infinite-width shallow neural network.

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SEMIPARAMETRIC INFERENCE BASED ON ADAPTIVELY COLLECTED DATA

BY LICONG LIN[1,a], KOULIK KHAMARU[2,b] AND MARTIN J. WAINWRIGHT[3,c]

[1]*Department of Statistics, University of California, Berkeley,* [a]*liconglin@berkeley.edu*

[2]*Department of Statistics, Rutgers University,* [b]*kk1241@stat.rutgers.edu*

[3]*Departments of Electrical Engineering and Computer Science and Mathematics, Massachusetts Institute of Technology,* [c]*wainwrigwork@gmail.com*

Many standard estimators, when applied to adaptively collected data, fail to be asymptotically normal, thereby complicating the construction of confidence intervals. We address this challenge in a semiparametric context: estimating the parameter vector of a generalized linear regression model contaminated by a nonparametric nuisance component. We construct suitably weighted estimating equations that account for adaptivity in data collection and provide conditions under which the associated estimates are asymptotically normal. Our results characterize the degree of "explorability" required for asymptotic normality to hold. For the simpler problem of estimating a linear functional, we provide similar guarantees under much weaker assumptions. We illustrate our general theory with concrete consequences for various problems, including standard linear bandits and sparse generalized bandits, and compare with other methods via simulation studies.

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THE NUMERAIRE E-VARIABLE AND REVERSE INFORMATION PROJECTION

BY MARTIN LARSSON[1,a], AADITYA RAMDAS[2,b] AND JOHANNES RUF[3,c]

[1]*Department of Mathematical Sciences, Carnegie Mellon University,* [a]*larsson@cmu.edu*
[2]*Departments of Statistics and ML, Carnegie Mellon University,* [b]*aramdas@cmu.edu*
[3]*Department of Mathematics, London School of Economics,* [c]*j.ruf@lse.ac.uk*

We consider testing a composite null hypothesis $\mathcal{P}$ against a point alternative Q using e-variables, which are nonnegative random variables X such that $\mathbb{E}_{\mathsf{P}}[X] \le 1$ for every $\mathsf{P} \in \mathcal{P}$. This paper establishes a fundamental result: under no conditions whatsoever on $\mathcal{P}$ or Q, there exists a special e-variable X^* that we call the numeraire, which is strictly positive and satisfies $\mathbb{E}_{\mathsf{Q}}[X/X^*] \le 1$ for every other e-variable X. In particular, X^* is log-optimal in the sense that $\mathbb{E}_{\mathsf{Q}}[\log(X/X^*)] \le 0$. Moreover, X^* identifies a particular subprobability measure P^* via the density $d\mathsf{P}^*/d\mathsf{Q} = 1/X^*$. As a result, X^* can be seen as a generalized likelihood ratio of Q against $\mathcal{P}$. We show that P^* coincides with the reverse information projection (RIPr) when additional assumptions are made that are required for the latter to exist. Thus, P^* is a natural definition of the RIPr in the absence of any assumptions on $\mathcal{P}$ or Q. In addition to the abstract theory, we provide several tools for finding the numeraire and RIPr in concrete cases. We discuss several nonparametric examples where we can indeed identify the numeraire and RIPr, despite not having a reference measure. Our results have interpretations outside of testing in that they yield the optimal Kelly bet against $\mathcal{P}$ if we believe reality follows Q. We end with a more general optimality theory that goes beyond the ubiquitous logarithmic utility. We focus on certain power utilities, leading to reverse Rényi projections in place of the RIPr, which also always exist.

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A DUALITY FRAMEWORK FOR ANALYZING RANDOM FEATURE AND TWO-LAYER NEURAL NETWORKS

BY HONGRUI CHEN[1,a], JIHAO LONG[2,b] AND LEI WU[3,c]

[1]*Department of Mathematics, Stanford University,* [a]*hongrui@stanford.edu*
[2]*Institute for Advanced Algorithms Research, Shanghai,* [b]*longjh1998@gmail.com*
[3]*School of Mathematical Sciences, Peking University,* [c]*leiwu@math.pku.edu.cn*

We consider the problem of learning functions within the $\mathcal{F}_{p,\pi}$ and Barron spaces, which play crucial roles in understanding random feature models (RFMs), two-layer neural networks as well as kernel methods. Leveraging tools from information-based complexity (IBC), we establish a dual equivalence between approximation and estimation and then apply it to study the learning of the preceding function spaces. The duality allows us to focus on the more tractable problem between approximation and estimation. To showcase the efficacy of our duality framework, we delve into two important but under-explored problems:

(1) *Random feature learning beyond kernel regime.* We derive sharp bounds for learning $\mathcal{F}_{p,\pi}$ using RFMs. Notably, the learning is efficient without the curse of dimensionality for $p > 1$. This underscores the extended applicability of RFMs beyond the traditional kernel regime, since $\mathcal{F}_{p,\pi}$ with $p < 2$ is strictly larger than the corresponding reproducing kernel Hilbert space (RKHS) where $p = 2$.

(2) *The L^∞ learning of RKHS.* We establish sharp, spectrum-dependent characterizations for the convergence of L^∞ learning error in both noiseless and noisy settings. Surprisingly, we show that popular kernel ridge regression can achieve near-optimal performance in L^∞ learning, despite it primarily minimizing square loss.

To establish the aforementioned duality, we introduce a type of IBC, termed I-complexity, to measure the size of a function class. Notably, I-complexity offers a tight characterization of learning in noiseless settings, yields lower bounds comparable to Le Cam's in noisy settings, and is versatile in deriving upper bounds. We believe that our duality framework holds potential for broad application in learning analysis across more scenarios.

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BELIEF IN DEPENDENCE: LEVERAGING ATOMIC LINEARITY IN DATA BITS FOR RETHINKING GENERALIZED LINEAR MODELS

BY BENJAMIN BROWN[1,a], KAI ZHANG[1,b] AND XIAO-LI MENG[2,c]

[1]*Department of Statistics and Operations Research, University of North Carolina at Chapel Hill,* [a]*brownb1@live.unc.edu,* [b]*zhangk@email.unc.edu*

[2]*Department of Statistics, Harvard University,* [c]*meng@stat.harvard.edu*

Two linearly uncorrelated binary variables must be also independent because nonlinear dependence cannot manifest with only two possible states. This inherent linearity is the atom of dependency constituting any complex form of relationship. Inspired by this observation, we develop a framework called binary expansion linear effect (BELIEF) for understanding arbitrary relationships with a binary outcome. Models from the BELIEF framework are easily interpretable because they describe the association of binary variables in the language of linear models, yielding convenient theoretical insight and striking Gaussian parallels. With BELIEF, one may study generalized linear models (GLM) through transparent linear models, providing insight into how the choice of link affects modeling. For example, setting a GLM interaction coefficient to zero does not necessarily lead to the kind of no-interaction model assumption as understood under their linear model counterparts. Furthermore, for a binary response, maximum likelihood estimation for GLMs paradoxically fails under complete separation, when the data are most discriminative, whereas BELIEF estimation automatically reveals the perfect predictor in the data that is responsible for complete separation. We explore these phenomena and provide related theoretical results. We also provide preliminary empirical demonstration of some theoretical results.

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SPARSITY MEETS CORRELATION IN GAUSSIAN SEQUENCE MODEL

BY SUBHODH KOTEKAL[a] AND CHAO GAO[b]

Department of Statistics, University of Chicago, [a]*skotekal@uchicago.edu,* [b]*chaogao@uchicago.edu*

We study estimation of an s-sparse signal in the p-dimensional Gaussian sequence model with equicorrelated observations and derive the minimax rate. A new phenomenon emerges from correlation, namely, the rate scales with respect to $p-2s$ and exhibits a phase transition at $p-2s \asymp \sqrt{p}$. Correlation is shown to be a blessing, provided it is sufficiently strong and the critical correlation level exhibits a delicate dependence on the sparsity level. Due to correlation, the minimax rate is driven by two subproblems: estimation of a linear functional (the average of the signal) and estimation of the signal's $(p-1)$-dimensional projection onto the orthogonal subspace. The high-dimensional projection is estimated via sparse regression, and the linear functional is cast as a robust location estimation problem. Existing robust estimators turn out to be suboptimal, and we show a kernel mode estimator with a widening bandwidth exploits the Gaussian character of the data to achieve the optimal estimation rate.

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CONFORMAL INFERENCE FOR RANDOM OBJECTS

BY HANG ZHOU[a] AND HANS-GEORG MÜLLER[b]

Department of Statistics, University of California, Davis, [a]*hgzhou@ucdavis.edu,* [b]*hgmueller@ucdavis.edu*

We develop an inferential tool kit for analyzing object-valued responses, which correspond to data situated in general metric spaces, paired with Euclidean predictors within the conformal framework. To this end, we introduce conditional profile average transport costs, where we compare distance profiles that correspond to one-dimensional distributions of probability mass falling into balls of increasing radius through the optimal transport cost when moving from one distance profile to another. The average transport cost to transport a given distance profile to all others is crucial for statistical inference in metric spaces and underpins the proposed conditional profile scores. A key feature of the proposed approach is to utilize the distribution of conditional profile average transport costs as conformity score for general metric space-valued responses, which facilitates the construction of prediction sets by the split conformal algorithm. We derive the uniform convergence rate of the proposed conformity score estimators and establish asymptotic conditional validity for the prediction sets. The finite sample performance for synthetic data in various metric spaces demonstrates that the proposed conditional profile score outperforms existing methods in terms of both coverage level and size of the resulting prediction sets, even in the special case of scalar Euclidean responses. We also demonstrate the practical utility of conditional profile scores for network data from New York taxi trips and for compositional data reflecting energy sourcing of U.S. states.

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STATISTICAL ALGORITHMS FOR LOW-FREQUENCY DIFFUSION DATA: A PDE APPROACH

BY MATTEO GIORDANO[1,a] AND SVEN WANG[2,b]

[1]*ESOMAS Department, University of Turin,* [a]*matteo.giordano@unito.it*
[2]*Department of Mathematics, Humboldt University Berlin,* [b]*sven.wang@hu-berlin.de*

We consider the problem of making nonparametric inference in a class of multi-dimensional diffusions in divergence form, from low-frequency data. Statistical analysis in this setting is notoriously challenging due to the intractability of the likelihood and its gradient, and computational methods have thus far largely resorted to expensive simulation-based techniques. In this article, we propose a new computational approach which is motivated by PDE theory and is built around the characterisation of the transition densities as solutions of the associated heat (Fokker-Planck) equation. Employing optimal regularity results from the theory of parabolic PDEs, we prove a novel characterisation for the gradient of the likelihood. Using these developments, for the nonlinear inverse problem of recovering the diffusivity, we then show that the numerical evaluation of the likelihood and its gradient can be reduced to standard elliptic eigenvalue problems, solvable by powerful finite element methods. This enables the efficient implementation of a large class of popular statistical algorithms, including (i) preconditioned Crank-Nicolson and Langevin-type methods for posterior sampling, and (ii) gradient-based descent optimisation schemes to compute maximum likelihood and maximum-a-posteriori estimates. We showcase the effectiveness of these methods via extensive simulation studies in a nonparametric Bayesian model with Gaussian process priors, in which both the proposed optimisation and sampling schemes provide good numerical recovery.

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MINIMAX RATE FOR MULTIVARIATE DATA UNDER COMPONENTWISE LOCAL DIFFERENTIAL PRIVACY CONSTRAINTS

BY CHIARA AMORINO[1,a] AND ARNAUD GLOTER[2,b]

[1]*Department of Mathematics, Université du Luxembourg,* [a]*chiara.amorino@upf.edu*
[2]*Laboratoire de Mathématiques et Modélisation d'Evry, CNRS, Univ Evry, Université Paris-Saclay,* [b]*arnaud.gloter@univ-evry.fr*

Our research analyses the balance between maintaining privacy and preserving statistical accuracy when dealing with multivariate data that is subject to *componentwise local differential privacy* (CLDP). With CLDP, each component of the private data is made public through a separate privacy channel. This allows for varying levels of privacy protection for different components or for the privatization of each component by different entities, each with their own distinct privacy policies. It also covers the practical situations where it is impossible to privatize jointly all the components of the raw data. We develop general techniques for establishing minimax bounds that shed light on the statistical cost of privacy in this context, as a function of the privacy levels $\alpha_1, \dots, \alpha_d$ of the d components.

We demonstrate the versatility and efficiency of these techniques by presenting various statistical applications. Specifically, we examine nonparametric density and joint moments estimation under CLDP, providing upper and lower bounds that match up to constant factors, as well as an associated data-driven adaptive procedure. Additionally, we conduct a detailed analysis of the effective privacy level, exploring how information about a private characteristic of an individual may be inferred from the publicly visible characteristics of the same individual.

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STRONG APPROXIMATIONS FOR EMPIRICAL PROCESSES INDEXED BY LIPSCHITZ FUNCTIONS

BY MATIAS D. CATTANEO[a] AND RUIQI RAE YU[b]

Department of Operations Research and Financial Engineering, Princeton University, [a]*cattaneo@princeton.edu,* [b]*rae.yu@princeton.edu*

This paper presents new uniform Gaussian strong approximations for empirical processes indexed by classes of functions based on d-variate random vectors ($d \geq 1$). First, a uniform Gaussian strong approximation is established for general empirical processes indexed by possibly Lipschitz functions, improving on previous results in the literature. In the setting considered by Rio (*Probab. Theory Related Fields* **98** (1994) 21–45), and if the function class is Lipschitzian, our result improves the approximation rate $n^{-1/(2d)}$ to $n^{-1/\max\{d,2\}}$, up to a polylog(n) term, where n denotes the sample size. Remarkably, we establish a valid uniform Gaussian strong approximation at the rate $n^{-1/2}\log n$ for $d = 2$, which was previously known to be valid only for univariate ($d = 1$) empirical processes via the celebrated Hungarian construction (Komlós, Major and Tusnády, *Z. Wahrsch. Verw. Gebiete* **32** (1975) 111–131). Second, a uniform Gaussian strong approximation is established for multiplicative separable empirical processes indexed by possibly Lipschitz functions, which addresses some outstanding problems in the literature (Chernozhukov, Chetverikov and Kato, *Ann. Statist.* **42** (2014) 1564–1597, Section 3). Finally, two other uniform Gaussian strong approximation results are presented when the function class is a sequence of Haar basis based on quasi-uniform partitions. Applications to nonparametric density and regression estimation are discussed.

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TESTING STATIONARITY AND CHANGE POINT DETECTION IN REINFORCEMENT LEARNING

BY MENGBING LI[1,a], CHENGCHUN SHI[2,c], ZHENKE WU[1,b] AND PIOTR FRYZLEWICZ[2,d]

[1]*Department of Biostatistics, University of Michigan,* [a]*mengbing@umich.edu,* [b]*zhenkewu@umich.edu*
[2]*Department of Statistics, London School of Economics and Political Science,* [c]*c.shi7@lse.ac.uk,* [d]*p.fryzlewicz@lse.ac.uk*

We consider reinforcement learning (RL) in possibly nonstationary environments. Many existing RL algorithms in the literature rely on the stationarity assumption that requires the state transition and reward functions to be constant over time. However, this assumption is restrictive in practice and is likely to be violated in a number of applications, including traffic signal control, robotics and mobile health. In this paper, we develop a model-free test to assess the stationarity of the optimal Q-function based on pre-collected historical data, without additional online data collection. Based on the proposed test, we further develop a change point detection method that can be naturally coupled with existing state-of-the-art RL methods designed in stationary environments for online policy optimization in nonstationary environments. The usefulness of our method is illustrated by theoretical results, simulation studies, and a real data example from the 2018 Intern Health Study. A Python implementation of the proposed procedure is publicly available at https://github.com/limengbinggz/CUSUM-RL.

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ADAPTIVE ESTIMATION OF THE $\mathbb{L}_2$-NORM OF A PROBABILITY DENSITY AND RELATED TOPICS I. LOWER BOUNDS

BY G. CLEANTHOUS[1,a], A. G. GEORGIADIS[2,b] AND O. V. LEPSKI[3,c]

[1]*Department of Mathematics and Statistics, Maynooth University,* [a]*galatia.cleanthous@mu.ie*
[2]*School of Computer Science and Statistics, Trinity College of Dublin,* [b]*georgiaa@tcd.ie*
[3]*Institut de Mathématique de Marseille, CNRS, Aix-Marseille Université,* [c]*oleg.lepski@univ-amu.fr*

We deal with the problem of the adaptive estimation of the $\mathbb{L}_2$–norm of a probability density on $\mathbb{R}^d$, $d \geq 1$, from independent observations. The unknown density is assumed to be uniformly bounded and to belong to the union of balls in the isotropic/anisotropic Nikolskii's spaces. We will show that the optimally adaptive estimators over the collection of considered functional classes do no exist. Also, in the framework of an abstract density model we present several generic lower bounds related to the adaptive estimation of an arbitrary functional of a probability density. These results having independent interest have no analogue in the existing literature. In the companion paper (Cleanthous, Georgiadis and Lepski (2024)), we prove that established lower bounds are tight and provide with explicit construction of adaptive estimators of $\mathbb{L}_2$–norm of the density.

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ADAPTIVE ESTIMATION OF THE $\mathbb{L}_2$-NORM OF A PROBABILITY DENSITY AND RELATED TOPICS II. UPPER BOUNDS VIA THE ORACLE APPROACH

BY G. CLEANTHOUS[1,a], A.G. GEORGIADIS[2,b] AND O.V. LEPSKI[3,c]

[1]*Department of Mathematics and Statistics, Maynooth University,* [a]*galatia.cleanthous@mu.ie*
[2]*School of Computer Science and Statistics, Trinity College of Dublin,* [b]*georgiaa@tcd.ie*
[3]*Institut de Mathématique de Marseille, CNRS, Aix-Marseille Université,* [c]*oleg.lepski@univ-amu.fr*

This is the second part of the research project initiated in Cleanthous, Georgiadis and Lepski (2024a). We deal with the problem of the adaptive estimation of the $\mathbb{L}_2$-norm of a probability density on $\mathbb{R}^d$, $d \geq 1$, from independent observations. The unknown density is assumed to be uniformly bounded by unknown constant and to belong to the union of balls in the isotropic/anisotropic Nikolskii's spaces. In Cleanthous, Georgiadis and Lepski (2024a), we have proved that the optimally adaptive estimators do no exist in the considered problem and provided with several lower bounds for the adaptive risk. In this part, we show that these bounds are tight and present the adaptive estimator which is obtained by a data-driven selection from a family of kernel-based estimators. The proposed estimation procedure as well as the computation of its risk are heavily based on new concentration inequalities for decoupled U-statistics of order two established in Section 4. It is also worth noting that all our results are derived from the unique oracle inequality which may be of independent interest.

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ASYMPTOTIC DISTRIBUTION OF MAXIMUM LIKELIHOOD ESTIMATOR IN GENERALIZED LINEAR MIXED MODELS WITH CROSSED RANDOM EFFECTS

BY JIMING JIANG[1,a]

[1] *Department of Statistics, University of California, Davis,* [a] *jimjiang@ucdavis.edu*

Generalized linear mixed models (GLMM) with crossed random effects is infamously known to present major challenges not only computationally but also theoretically. In fact, to date only consistency of the maximum likelihood estimators (MLE) has been proved for GLMM with crosses random effects. We introduce a new technique in asymptotic analysis built on a second-order Laplace approximation (LA) of a conditional expectation, whose coefficients are carefully evaluated. The LA leads to a large system of equations involving the conditional expectations, which is asymptotically inverted to obtain explicit approximation to the conditional expectations. This powerful technique leads to a new approach to establishing asymptotic distribution of the MLE when the likelihood function is intractable. As a result, asymptotic normality of the MLE is rigorously established for GLMM with crossed random effects, bringing answer to an open problem dating back to decades ago.

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SELF-NORMALIZED CRAMÉR TYPE MODERATE DEVIATION THEOREM FOR GAUSSIAN APPROXIMATION

BY JINGKUN QIU[1,a], SONG XI CHEN[2,b] AND QI-MAN SHAO[3,c]

[1]*Guanghua School of Management, Peking University,* [a]*jkqiu@stu.pku.edu.cn*

[2]*Department of Statistics and Data Science, Tsinghua University,* [b]*sxchen@tsinghua.edu.cn*

[3]*Department of Statistics and Data Science, Shenzhen International Center for Mathematics, National Center for Applied Mathematics Shenzhen, Southern University of Science and Technology,* [c]*shaoqm@sustech.edu.cn*

Berry–Esseen type bounds for Gaussian approximation of standardized sums have been extensively studied under exponential type moment conditions. In this paper, a Cramér type moderate deviation theorem is established for self-normalized Gaussian approximation under finite moment conditions. More specifically, let $X_1, X_2, \dots, X_n$ be i.i.d. $\mathbb{R}^p$-valued random vectors with zero means. Let $S_{n,j} = \sum_{i=1}^n X_{ij}$ and $V_{n,j}^2 = \sum_{i=1}^n X_{ij}^2$. We show that if the correlation matrix of X_1 is I_p and the third moment of X_1 is finite, then

$$\frac{\mathbb{P}(\max_{1\le j\le p} S_{n,j}/V_{n,j} > x)}{\mathbb{P}(\max_{1\le j\le p} Z_j > x)} \to 1$$

uniformly for $0 \le x \le o(n^{1/6})$ and for all $p \ge 1$, where $Z_1, \dots, Z_p$ are independent standard normal random variables. Similar result is also established for large x when X_1 has a general correlation matrix. The proof is based on a new Cramér type moderate deviation theorem for the minimum of several self-normalized sums. As an application, we propose a high dimensional one-sample t-test that allows for an exponential growth of p without requiring the commonly used sub-Gaussian assumption.

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SEMIPARAMETRIC ADAPTIVE ESTIMATION UNDER INFORMATIVE SAMPLING

BY KOSUKE MORIKAWA[1,3,a], YOSHIKAZU TERADA[1,2,b] AND JAE KWANG KIM[3,c]

[1]*Graduate School of Engineering Science, Osaka University,* [a]*morikawa@iastate.edu*
[2]*Center for Advanced Integrated Intelligence Research, RIKEN,* [b]*yoshikazu.terada.es@osaka-u.ac.jp*
[3]*Department of Statistics, Iowa State University,* [c]*jkim@iastate.edu*

In probability sampling, sampling weights are often used to remove selection bias in the sample. The Horvitz–Thompson estimator is well known to be consistent and asymptotically normally distributed; however, it is not necessarily efficient. This study derives the semiparametric efficiency bound for various target parameters by considering the survey weights as random variables and consequently proposes two semiparametric estimators with working models on the survey weights. One estimator assumes a reasonable parametric working model, but the other estimator does not require specific working models by using the debiased/double machine learning method. The proposed estimators are consistent, asymptotically normal, and efficient in a class of regular and asymptotically linear estimators. A limited simulation study is conducted to investigate the finite sample performance of the proposed method. The proposed method is applied to the 1999 Canadian Workplace and Employee Survey data.

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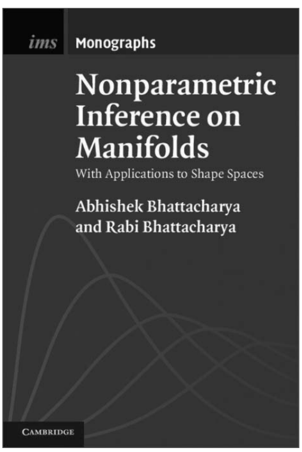

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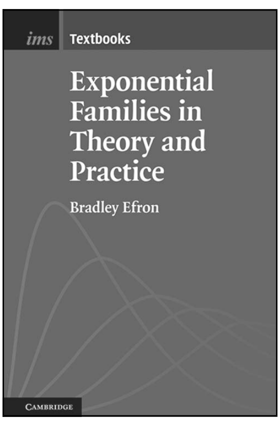

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