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RANK TESTS FOR PCA UNDER WEAK IDENTIFIABILITY

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In a triangular array framework where n observations are randomly sampled from a p-dimensional elliptical distribution with shape matrix $\mathbf{V}_n$, we consider the problem of testing the null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against the alternative hypothesis $\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, where $\boldsymbol{\theta}$ is the (fixed) leading unit eigenvector of $\mathbf{V}_n$ and $\boldsymbol{\theta}_0$ is a given unit p-vector. The dependence of the shape matrix on the sample size allows us to consider challenging asymptotic scenarios in which the parameter of interest $\boldsymbol{\theta}$ is unidentified in the limit, because the ratio between both leading eigenvalues of $\mathbf{V}_n$ converges to one. We carefully study the corresponding limiting experiments under such *weak identifiability*, and we show that these may be LAN or non-LAN. While earlier work in the framework was strictly limited to Gaussian distributions, where the study of local log-likelihood ratios could simply rely on explicit expressions, our asymptotic investigation allows for essentially arbitrary elliptical distributions. This requires original results on quadratic mean differentiable families for triangular arrays of observations, which are likely to be of interest in other models, too. Even in non-LAN experiments, our results enable us to investigate, through Le Cam's first and third lemmas, the asymptotic null and nonnull properties of multivariate rank tests. These nonparametric tests are shown to exhibit an excellent behavior under weak identifiability: not only do they maintain the target nominal size irrespective of the amount of weak identifiability, but they also keep their outstanding uniform efficiency properties under such nonstandard scenarios. In particular, Gaussian-score rank tests, under arbitrarily weak identifiability, still *uniformly* dominate their parametric pseudo-Gaussian competitor in terms of asymptotic relative efficiencies. Our theoretical results, which are the first ones to study rank tests in the triangular array framework allowing for weak identifiability, are supported by several Monte Carlo exercises.

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DISTRIBUTIONALLY ROBUST LEARNING FOR MULTISOURCE UNSUPERVISED DOMAIN ADAPTATION

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Empirical risk minimization often performs poorly when the distribution of the target domain differs from those of the source domains. To address such potential distributional shifts, we develop an unsupervised domain adaptation approach that leverages labeled data from multiple source domains and unlabeled data from the target domain. We introduce a distributionally robust model that optimizes an adversarial reward based on explained variance across a class of target distributions, ensuring generalization to the target domain. We show that the proposed robust model is a weighted average of conditional outcome models from the source domains. This formulation allows us to compute the robust model through the aggregation of source models, which can be estimated using various machine learning algorithms of the user's choice such as random forests, boosting and neural networks. Additionally, we introduce a bias-correction step to obtain a more accurate aggregation weight, which is effective for various machine learning algorithms. Our framework can be interpreted as a distributionally robust federated learning approach that satisfies privacy constraints while providing insights into the importance of each source for prediction on the target domain. The performance of our method is evaluated on both simulated and real data.

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TRACE TEST FOR HIGH-DIMENSIONAL COINTEGRATION

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This paper studies Johansen's (*J. Econom. Dynam. Control* **12** (1988) 231–254) trace test for cointegration in high-dimensional data. We show that when both cross-sectional and temporal dimension of the data go to infinity proportionally, the shifted and scaled modified trace statistic converges to a Gaussian random variable. We give explicit formulae for the shift and scale parameters as well as for the mean and variance of the Gaussian limit. Monte Carlo analysis shows excellent size properties of the asymptotic test, which is an improvement over the Bartlett-corrected versions of the original trace test, especially for relatively large ratios of the dimensionality to the sample size. The Monte Carlo also reveals a nonmonotonicity of the power of the test. We comment on the source of such a nonmonotonicity.

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ESTIMATION OF GROUPED TIME-VARYING NETWORK VECTOR AUTOREGRESSIVE MODELS

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This paper introduces a flexible time-varying network vector autoregressive model framework for large-scale time series. A latent group structure is imposed on the heterogeneous and node-specific time-varying momentum and network spillover effects so that the number of unknown time-varying coefficients to be estimated can be reduced considerably. A classic agglomerative clustering algorithm with nonparametrically estimated distance matrix is combined with a ratio criterion to consistently estimate the latent group number and membership. A postgrouping local linear smoothing method is proposed to estimate the group-specific time-varying momentum and network effects, substantially improving the convergence rates of the preliminary estimates which ignore the latent structure. We further modify the methodology and theory to allow for structural breaks in either the group membership, group number or group-specific coefficient functions. Numerical studies including Monte-Carlo simulation and an empirical application are presented to examine the finite-sample performance of the developed model and methodology.

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LARGE-SCALE MULTIPLE TESTING: FUNDAMENTAL LIMITS OF FALSE DISCOVERY RATE CONTROL AND COMPOUND ORACLE

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The false discovery rate (FDR) and the false nondiscovery rate (FNR), defined as the expected false discovery proportion (FDP) and the false nondiscovery proportion (FNP), are the most popular benchmarks for multiple testing. Despite the theoretical and algorithmic advances in recent years, the optimal trade-off between the FDR and the FNR has been largely unknown, except for certain restricted classes of decision rules, for example, separable rules, or for other performance metrics, for example, the marginal FDR and the marginal FNR (mFDR and mFNR). In this paper we determine the asymptotically optimal FDR-FNR trade-off under the two-group random mixture model when the number of hypotheses tends to infinity. Distinct from the optimal mFDR-mFNR trade-off, which is achieved by separable decision rules, the optimal FDR-FNR trade-off requires compound rules, even in the large-sample limit and for models as simple as the Gaussian location model. This suboptimality of separable rules also holds for other objectives, such as maximizing the expected number of true discoveries. Finally, to address the limitation of the FDR, which only controls the expectation but not the fluctuation of the FDP, we also determine the optimal tradeoff when the FDP is controlled with high probability and show it coincides with that of the mFDR and the mFNR. Extensions to models with a fixed nonnull proportion are also obtained.

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ANALYSIS OF SINGULAR SUBSPACES UNDER RANDOM PERTURBATIONS

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We present a comprehensive analysis of singular vector and singular subspace perturbations in the signal-plus-noise matrix model with random Gaussian noise. Assuming a low-rank signal matrix, we extend the Davis–Kahan–Wedin theorem in a fully generalized manner, applicable to any unitarily invariant matrix norm, building on previous results by O'Rourke, Vu, and the author. Our analysis provides fine-grained insights, including ℓ_∞ bounds for singular vectors, $\ell_{2,\infty}$ bounds for singular subspaces, and results for linear and bilinear functions of singular vectors. Additionally, we derive $\ell_{2,\infty}$ bounds on perturbed singular vectors, taking into account the weighting by their corresponding singular values. Finally, we explore practical implications of these results in the Gaussian mixture model and the submatrix localization problem.

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VERSATILE DIFFERENTIALLY PRIVATE LEARNING FOR GENERAL LOSS FUNCTIONS

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This paper aims to provide a versatile privacy-preserving release mechanism along with a unified approach for subsequent parameter estimation and statistical inference. We propose a privacy mechanism based on zero-inflated symmetric multivariate Laplace (ZIL) noise, which requires no prior specification of subsequent analysis tasks, allows for general loss functions under minimal conditions, imposes no limit on the number of analyses, and is adaptable to increasing data volume in online scenarios. We derive the trade-off function for the proposed ZIL mechanism, which characterizes its privacy protection level. Furthermore, to formalize the local differential privacy (LDP) property of the ZIL mechanism, we extend the classical ε-LDP to a more general f-LDP framework. To address scenarios where only individual attribute values require protection, we propose attribute-level differential privacy (ADP) and its local version. Within the M-estimation framework, we introduce a novel doubly random (DR) corrected loss for the ZIL mechanism, which yields consistent and asymptotically normal M-estimates under differential privacy constraints. The proposed approach is computationally efficient and does not require numerical integration or differentiation for noisy data. It applies to a broad class of loss functions, including nonsmooth ones. Two alternative estimators for smooth loss are also proposed with asymptotic properties. The cost of privacy in terms of estimation efficiency for these three estimators is evaluated both theoretically and numerically.

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OPTIMAL EIGENVALUE SHRINKAGE IN THE SEMICIRCLE LIMIT

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In response to the increasing dimensionality of modern datasets, recent theoretical studies of covariance estimation frequently adopt the *proportional-growth* asymptotic framework in which the sample size n and dimension p are comparable, with $n, p \to \infty$ and $\gamma_n := p/n \to \gamma > 0$. However, in many datasets—perhaps most—the sample size and dimension are highly imbalanced. To address this, we consider the *disproportional-growth* asymptotic framework, where $n, p \to \infty$ and $\gamma_n \to 0$ or $\gamma_n \to \infty$. These regimes give rise to novel behavior distinct from those in the proportional-growth and classical fixed-p settings.

We work under the spiked covariance model in which the theoretical covariance matrix is a low-rank perturbation of the identity. For each of 15 loss functions, we derive closed-form optimal shrinkage and thresholding rules; for several losses, optimality takes the particularly strong form of *unique asymptotic admissibility*. These optimal procedures involve substantial eigenvalue shrinkage and yield significant improvements over the standard empirical covariance estimator.

Practitioners may ask whether their data is better modeled under the proportional or disproportional frameworks and which of the corresponding procedures to apply. Fortunately, it is possible to remain framework-agnostic: a single, unified set of shrinkage rules—depending only on the aspect ratio γ_n of the given data—achieves asymptotic optimality in both regimes.

At the heart of these phenomena is the spiked Wigner model in which a low-rank matrix is perturbed by symmetric noise. Under both the (scaled) spiked covariance model as $\gamma_n \to 0$ and the spiked Wigner model, the empirical spectral distributions converge to the semicircle law. Exploiting this connection, we derive optimal spiked-Wigner shrinkage rules, which are of independent and fundamental interest.

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SPECTRUM-AWARE DEBIASING: A MODERN INFERENCE FRAMEWORK WITH APPLICATIONS TO PRINCIPAL COMPONENTS REGRESSION

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Debiasing is a fundamental concept in high-dimensional statistics. While degrees-of-freedom adjustment is the state-of-the-art debiasing technique in high-dimensional linear regression, it largely remains limited to independent, identically distributed samples and sub-Gaussian covariates. These limitations hinder its wider practical use. In this paper we break this barrier and introduce Spectrum-Aware Debiasing—a novel inference method that applies to challenging high-dimensional regression problems with structured row-column dependencies, heavy tails, asymmetric properties, and latent low-rank structures. Our method achieves debiasing through a rescaled gradient descent step, where the rescaling factor is derived from the spectral properties of the sample covariance matrix. This spectrum-based approach enables accurate debiasing in much broader contexts. We study the common modern regime where the number of features and samples scale proportionally. We establish asymptotic normality of our proposed estimator (suitably centered and scaled) under various convergence notions when the covariates are right-rotationally invariant. We further prove a spectral universality result, extending our guarantees to a much broader class of covariate distributions. Furthermore, we devise a consistent estimator for the asymptotic variance.

Our work has two notable by-products: First, Spectrum-Aware Debiasing rectifies the bias in principal components regression (PCR), providing the first debiased PCR estimator in high dimensions. Second, we introduce a principled test for checking the presence of alignment between the signal and the eigenvectors of the sample covariance matrix. This test is independently valuable for statistical methods developed using approximate message passing, leave-one-out, random matrix theory, or convex Gaussian min-max theorems. We demonstrate the utility of our method through diverse simulated and real data experiments.

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SCALABLE INFERENCE FOR NONPARAMETRIC STOCHASTIC APPROXIMATION IN REPRODUCING KERNEL HILBERT SPACES

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Stochastic approximation (SA) is a powerful and scalable computational method for iteratively estimating the solution of optimization problems in the presence of randomness, particularly well suited for large-scale and streaming data settings. In this work we propose a theoretical framework for stochastic approximation (SA) applied to nonparametric least squares in reproducing kernel Hilbert spaces (RKHS), enabling online statistical inference in nonparametric regression models. We achieve this by constructing asymptotically valid pointwise (and simultaneous) confidence intervals (bands) for local (and global) inference of the nonlinear regression function, via employing an online multiplier bootstrap approach to a functional stochastic gradient descent (SGD) algorithm in the RKHS. Our main theoretical contributions consist of a unified framework for characterizing the nonasymptotic behavior of the functional SGD estimator and demonstrating the consistency of the multiplier bootstrap method. The proof techniques involve the development of a higher-order expansion of the functional SGD estimator under the supremum norm metric and the Gaussian approximation of suprema of weighted and non-identically distributed empirical processes. Our theory specifically reveals an interesting relationship between the tuning of step sizes in SGD for estimation and the accuracy of uncertainty quantification.

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PRECISE ASYMPTOTICS OF BAGGING REGULARIZED M-ESTIMATORS

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We characterize the squared prediction risk of ensemble estimators obtained through subagging (subsample bootstrap aggregating) regularized M-estimators and construct a consistent estimator for the risk. Specifically, we consider a heterogeneous collection of $M \geq 1$ regularized M-estimators, each trained with (possibly different) subsample sizes, convex differentiable losses, and convex regularizers. We operate under the proportional asymptotics regime, where the sample size n, feature size p, and subsample sizes k_m for $m \in [M]$ all diverge with fixed limiting ratios n/p and k_m/n. Key to our analysis is a new result on the joint asymptotic behavior of correlations between the estimator and residual errors on overlapping subsamples, governed through a (provably) contractive nonlinear system of equations. Of independent interest we also establish convergence of trace functionals related to degrees of freedom in the nonensemble setting (with $M = 1$) along the way, extending previously known cases for squared loss with ridge and lasso regularizers. When specialized to homogeneous ensembles trained with a common loss, regularizer, and subsample size, the risk characterization sheds some light on the (implicitly) induced regularization effect due to the ensemble and subsample sizes (M, k). For any ensemble size M, optimally tuning subsample size yields samplewise monotonic risk. For the full-ensemble estimator (when $M \to \infty$), the optimal subsample size $k^\star$ tends to be in the overparameterized regime ($k^\star \leq \min\{n, p\}$), when explicit regularization is vanishing. Finally, joint optimization of subsample size, ensemble size, and regularization can significantly outperform regularizer optimization alone on the full data (without any subagging).

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FINITE- AND LARGE SAMPLE INFERENCE FOR MODEL AND COEFFICIENTS IN HIGH-DIMENSIONAL LINEAR REGRESSION WITH REPRO SAMPLES

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In this paper, we present a novel and effective inference approach to conduct both finite and large sample inference for high-dimensional linear regression models. This approach is developed under the so-called *repro samples* framework, in which we conduct statistical inference by creating and studying the behavior of artificial samples that are obtained by mimicking the sampling mechanism of the data. We construct confidence sets for (a) the true model corresponding to the nonzero coefficients, (b) a single or any collection of regression coefficients and (c) both the model and regression coefficients jointly. To facilitate the constructions of these confidence sets and overcome computational difficulties of searching all possible models, we use an innovative Fisher inversion technique to construct a model candidate set that includes the true sparse model with probability close to 1 for models with both Gaussian and non-Gaussian errors. The proposed approach fills in two major gaps in the high-dimensional regression literature: (1) lack of effective approaches to addressing model selection uncertainty and providing valid inference for the underlying true model; (2) lack of effective inference approaches to guarantee finite sample performance. We provide both finite sample and asymptotic results to theoretically guarantee the performance of the proposed methods. In addition, our numerical results demonstrate that the proposed methods are valid and achieve better coverage with smaller confidence sets than the current state-of-the-art approaches, such as debiasing and bootstrap approaches.

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INFERRING THE DEPENDENCE GRAPH DENSITY OF BINARY GRAPHICAL MODELS IN HIGH DIMENSION

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We consider a system of binary interacting chains describing the dynamics of a group of N components that, at each time unit, either send some signal to the others or remain silent otherwise. The interactions among the chains are encoded by a directed Erdös–Rényi random graph with unknown parameter $p \in (0, 1)$. Moreover, the system is structured within two populations (excitatory chains versus inhibitory ones), which are coupled via a mean field interaction on the underlying Erdös–Rényi graph. In this paper, we address the question of inferring the connectivity parameter p based only on the observation of the interacting chains over T time units. In our main result, we show that the connectivity parameter p can be estimated with rate $N^{-1/2} + N^{1/2}/T + (\log(T)/T)^{1/2}$ through an easy-to-compute estimator. Our analysis relies on a precise study of the spatiotemporal decay of correlations of the interacting chains. This is done through the study of coalescing random walks defining a backward regeneration representation of the system. Interestingly, we also show that this backward regeneration representation allows us to perfectly sample the system of interacting chains (conditionally on each realization of the underlying Erdös–Rényi graph) from its stationary distribution. These probabilistic results have an interest in its own.

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EIGENVECTOR OVERLAPS IN LARGE SAMPLE COVARIANCE MATRICES AND NONLINEAR SHRINKAGE ESTIMATORS

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Consider a data matrix $Y = [\mathbf{y}_1, \ldots, \mathbf{y}_N]$ of size $M \times N$, where the columns are independent observations from a random vector $\mathbf{y}$ with zero mean and population covariance Σ. Let $\mathbf{u}_i$ and $\mathbf{v}_j$ denote the left and right singular vectors of Y, respectively. This study investigates the eigenvector/singular vector overlaps $\langle \mathbf{u}_i, D_1 \mathbf{u}_j \rangle$, $\langle \mathbf{v}_i, D_2 \mathbf{v}_j \rangle$ and $\langle \mathbf{u}_i, D_3 \mathbf{v}_j \rangle$, where D_k are general deterministic matrices with bounded operator norms. In the high-dimensional regime, where the dimension M scales proportionally with the sample size N, we establish the convergence in probability of these eigenvector overlaps towards their deterministic counterparts with explicit convergence rates. Building upon these findings, we offer a more precise characterization of the loss associated with Ledoit and Wolf's nonlinear shrinkage estimators of the population covariance Σ.

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OBJECT DETECTION UNDER THE LINEAR SUBSPACE MODEL WITH APPLICATION TO CRYO-EM IMAGES

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Detecting multiple unknown objects in noisy data is a key problem in many scientific fields, such as electron microscopy imaging. A common model for the unknown objects is the linear subspace model, which assumes that the objects can be expanded in some known basis (such as the Fourier basis). In this paper, we develop an object detection algorithm that under the linear subspace model is asymptotically guaranteed to detect all objects, while controlling the familywise error rate or the false discovery rate. Numerical simulations show that the algorithm also controls the error rate with high power in the nonasymptotic regime, even in highly challenging regimes. We apply the proposed algorithm to an experimental electron microscopy data set, and show that it outperforms existing standard software.

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PCA FOR POINT PROCESSES

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We introduce a novel statistical framework for the analysis of replicated point processes that allows for the study of point pattern variability at a population level. By treating point process realizations as random measures, we adopt a functional analysis perspective and propose a form of functional Principal Component Analysis (fPCA) for point processes. The originality of our method is to base our analysis on the cumulative mass functions of the random measures, which gives us a direct and interpretable analysis. Key theoretical contributions include establishing a Karhunen–Loève expansion for the random measures and a Mercer theorem for covariance measures. We establish convergence in a strong sense, and introduce the concept of principal measures, which can be seen as latent processes governing the dynamics of the observed point patterns. We propose an easy-to-implement estimation strategy of eigenelements for which parametric rates are achieved. We fully characterize the solutions of our approach to Poisson and Hawkes processes and validate our methodology via simulations and diverse applications in seismology, single-cell biology and neurosiences, demonstrating its versatility and effectiveness. Our method is implemented in the `pppca` R-package.

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PARAMETER IDENTIFICATION IN LINEAR NON-GAUSSIAN CAUSAL MODELS UNDER GENERAL CONFOUNDING

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Linear non-Gaussian causal models postulate that each random variable is a linear function of parent variables and non-Gaussian exogenous error terms. We study identification of the linear coefficients when such models contain latent variables. Our focus is on the commonly studied acyclic setting, where each model corresponds to a directed acyclic graph (DAG). For this case, prior literature has demonstrated that connections to overcomplete independent component analysis yield effective criteria to decide parameter identifiability in latent variable models. However, this connection is based on the assumption that the observed variables linearly depend on the latent variables. Departing from this assumption, we treat models that allow for arbitrary nonlinear latent confounding. Our main result is a graphical criterion that is necessary and sufficient for deciding the generic identifiability of direct causal effects. Moreover, we provide an algorithmic implementation of the criterion with a run time that is polynomial in the number of observed variables. Finally, we report on estimation heuristics based on the identification result and explore a generalization to models with feedback loops.

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GENERALIZED MULTILINEAR MODELS FOR SUFFICIENT DIMENSION REDUCTION ON TENSOR-VALUED PREDICTORS

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We consider supervised learning problems with tensor-valued input. We derive multilinear sufficient reductions for the regression or classification problem by modeling the conditional distribution of the predictors given the response as a member of the quadratic exponential family. We develop estimation procedures of sufficient reductions for both continuous and binary tensor-valued predictors. We prove the consistency and asymptotic normality of the estimated sufficient reduction using manifold theory. For multilinear normal predictors, the estimation algorithm is highly computationally efficient and is also applicable to situations where the dimension of the reduction exceeds the sample size. Our method outperforms competing techniques in both simulated settings and real-world datasets involving continuous and binary tensor-valued predictors.

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THE DISTRIBUTIONALLY ROBUST PREDICTION ERROR OF THE √LASSO AND RELATED ESTIMATORS

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We study the classical problem of predicting an outcome variable, Y, using a linear combination of a d-dimensional covariate vector, $\mathbf{X}$. We are interested in linear predictors whose coefficients solve

$$\inf_{\boldsymbol{\beta}\in\mathbb{R}^d} \left(\mathbb{E}_{\mathbb{P}_n}\left[\left|Y-\mathbf{X}^\top\boldsymbol{\beta}\right|^r\right]\right)^{1/r} + \delta\rho(\boldsymbol{\beta}),$$

where $\delta > 0$ is a regularization parameter, $\rho : \mathbb{R}^d \to \mathbb{R}_+$ is a convex penalty function, $\mathbb{P}_n$ is the empirical distribution of the data and $r \geq 1$. Our main contribution is a new bound on the out-of-distribution prediction error of such estimators.

The new bound is obtained by combining three new sets of results. First, we provide conditions under which linear predictors based on these estimators solve a *distributionally robust optimization* problem: they minimize the worst-case prediction error over distributions that are close to each other in a type of *max-sliced Wasserstein metric*. Second, we provide a detailed finite-sample and asymptotic analysis of the statistical properties of the balls of distributions over which the worst-case prediction error is analyzed. Third, we present an oracle recommendation for the choice of regularization parameter, δ, that guarantees good out-of-distribution prediction error.

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GENERALIZED LINEAR SPECTRAL STATISTICS OF HIGH-DIMENSIONAL SAMPLE COVARIANCE MATRICES AND ITS APPLICATIONS

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In this paper, we introduce the **G**eneralized **L**inear **S**pectral **S**tatistics (GLSS) of a high-dimensional sample covariance matrix $\boldsymbol{S}_n$, denoted as $\operatorname{tr} f(\boldsymbol{S}_n)\boldsymbol{B}_n$, which effectively captures distinct spectral properties of $\boldsymbol{S}_n$ by incorporating an ancillary matrix $\boldsymbol{B}_n$ and a test function f. The joint asymptotic normality of GLSS associated with different test functions is established under mild assumptions on $\boldsymbol{B}_n$ and the underlying distribution, when the dimension n and sample size N are comparable. The convergence rate of GLSS is determined by $\sqrt{N/\operatorname{rank}(\boldsymbol{B}_n)}$. Subsequently, we propose a novel functional projection approach based on GLSS for hypothesis testing on eigenspaces of "population-spiked" covariance matrices, showcasing a universality phenomenon in the magnitude of the spikes. The theoretical accuracy of our results established for GLSS and the advantages of the newly suggested testing procedure are demonstrated through various numerical studies.

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REVIVING PSEUDO-INVERSES: ASYMPTOTIC PROPERTIES OF LARGE DIMENSIONAL MOORE–PENROSE AND RIDGE-TYPE INVERSES WITH APPLICATIONS

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In this paper, we derive high-dimensional asymptotic properties of the Moore–Penrose inverse and, as a byproduct, of various ridge-type inverses of the sample covariance matrix. In particular, the analytical expressions of the asymptotic behavior of the weighted sample trace moments of generalized inverse matrices are deduced in terms of the partial exponential Bell polynomials, which can be easily computed in practice. The existent results for pseudo-inverses are extended in several directions: (i) First, the population covariance matrix is not assumed to be a multiple of the identity matrix; (ii) Second, the assumption of normality is not used in the derivation; (iii) Third, the asymptotic results are derived under the high-dimensional asymptotic regime. Our findings provide universal methodology for construction of fully data-driven improved shrinkage estimators of the precision matrix, optimal portfolio weights and beyond. It is found that the Moore–Penrose inverse acts asymptotically as a certain regularizer of the true covariance matrix and it seems that its proper transformation (shrinkage) performs similar to or even outperforms the existing benchmarks in many applications, while keeping the computational time as minimal as possible.

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ADAPTIVE BAYESIAN REGRESSION ON DATA WITH LOW INTRINSIC DIMENSIONALITY

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We study how the posterior contraction rate under a Gaussian process (GP) prior depends on the intrinsic dimension of the predictors and the smoothness of the regression function. An open question is whether a generic GP prior that does not incorporate knowledge of the intrinsic lower-dimensional structure of the predictors can attain an adaptive rate for a broad class of such structures. We show that this is indeed the case, establishing conditions under which the posterior contraction rates become adaptive to the intrinsic dimension in terms of the covering number of the data domain (the Minkowski dimension) and prove the nonparametric posterior contraction rate, up to a logarithmic factor. When the domain is a compact manifold, we prove the RKHS approximation to intrinsically defined Hölder functions on the manifold of any order of smoothness by a novel analysis, leading to the optimal adaptive posterior contraction rate. We propose an empirical Bayes prior on the kernel bandwidth using kernel affinity and k-nearest neighbor statistics, bypassing explicit estimation of the intrinsic dimension. The efficiency of the proposed Bayesian regression approach is demonstrated in various numerical experiments.

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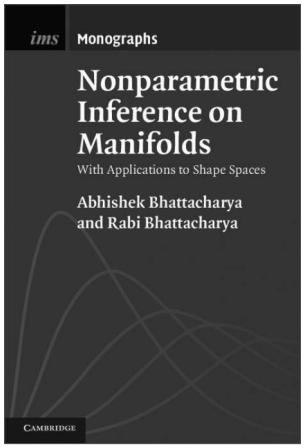

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