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SPARSE PCA: A NEW SCALABLE ESTIMATOR BASED ON INTEGER PROGRAMMING

BY KAYHAN BEHDIN[1,a] AND RAHUL MAZUMDER[1,2,b]

[1]*Operations Research Center, Massachusetts Institute of Technology*
[2]*Sloan School of Management, Massachusetts Institute of Technology,* [a]*behdink@mit.edu,* [b]*rahulmaz@mit.edu*

We consider the sparse principal component analysis (SPCA) problem under the well-known spiked covariance model. Recent work has shown that the SPCA problem can be reformulated as a mixed integer program (MIP) and can be solved to global optimality, leading to estimators that are known to enjoy optimal statistical properties. However, prior MIP algorithms for SPCA appear to be limited in terms of scalability to up to a thousand features or so. In this paper we propose a new estimator for SPCA which can be formulated as a MIP. Different from earlier work, we make use of the underlying spiked covariance model and properties of the multivariate Gaussian distribution to arrive at our estimator. We establish statistical guarantees for our proposed estimator in terms of estimation error and support recovery. We derive guarantees under departures from the spiked covariance model and for approximate solutions to the optimization problem. We propose a custom algorithm to solve the MIP, which scales better than off-the-shelf solvers, and demonstrate that our approach can be much more computationally attractive compared to earlier exact MIP-based approaches for the SPCA problem. Our numerical experiments on synthetic and real datasets show that our algorithms can address problems with up to 20,000 features in minutes; and generally result in favorable statistical properties compared to existing popular approaches for SPCA.

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ADAPTIVE ROBUST CONFIDENCE INTERVALS

BY YUETIAN LUO[1,a] AND CHAO GAO[2,b]

[1]*Data Science Institute, University of Chicago,* [a]*yuetian@uchicago.edu*
[2]*Department of Statistics, University of Chicago,* [b]*chaogao@uchicago.edu*

This paper studies the construction of adaptive confidence intervals under Huber's contamination model when the contamination proportion is unknown. For the robust confidence interval of a Gaussian mean, we show that the optimal length of an adaptive interval must be exponentially wider than that of a nonadaptive one. An optimal construction is achieved through simultaneous uncertainty quantification of quantiles at all levels. The results are further extended beyond the Gaussian location model by addressing a general family of robust hypothesis testing. In contrast to adaptive robust estimation, our findings reveal that the optimal length of an adaptive robust confidence interval critically depends on the distribution's shape.

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ONLINE TENSOR LEARNING: COMPUTATIONAL AND STATISTICAL TRADE-OFFS, ADAPTIVITY AND OPTIMAL REGRET

BY JINGYANG LI[1,a], JIAN-FENG CAI[2,b], YANG CHEN[3,d] AND DONG XIA[2,c]

[1]*Department of Statistics and Data Science, School of Management, Fudan University,* [a]*jjyyli.acad@gmail.com*
[2]*Department of Mathematics, Hong Kong University of Science and Technology,* [b]*jfcai@ust.hk,* [c]*madxia@ust.hk*
[3]*Department of Statistics, University of Michigan,* [d]*ychenang@umich.edu*

Large tensor learning algorithms are typically computationally expensive and require storing a vast amount of data. In this paper we propose a unified online Riemannian gradient descent (oRGrad) algorithm for tensor learning, which is computationally efficient, consumes much less memory, and can handle sequentially arriving data while making timely predictions. The algorithm is applicable to both linear and generalized linear models. If the time horizon T is known, oRGrad achieves statistical optimality by choosing an appropriate fixed step size. We find that noisy tensor completion particularly benefits from online algorithms by avoiding the trimming procedure and ensuring sharp entrywise statistical error, which is often technically challenging for offline methods. The regret of oRGrad is analyzed, revealing a fascinating trilemma concerning the computational convergence rate, statistical error, and regret bound. By selecting an appropriate constant step size, oRGrad achieves an $O(T^{1/2})$ regret. We then introduce the adaptive-oRGrad algorithm, which can achieve the optimal $O(\log T)$ regret by adaptively selecting step sizes, regardless of whether the time horizon is known. The adaptive-oRGrad algorithm can attain a statistically optimal error rate without knowing the horizon. Comprehensive numerical simulations corroborate our theoretical findings. We show that oRGrad significantly outperforms its offline counterpart in predicting the solar F10.7 index with tensor predictors that monitor space weather impacts.

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STATISTICAL INFERENCE FOR LOW-RANK TENSORS: HETEROSKEDASTICITY, SUBGAUSSIANITY, AND APPLICATIONS

BY JOSHUA AGTERBERG[1,a] AND ANRU R. ZHANG[2,b]

[1]*Department of Statistics, University of Illinois Urbana-Champaign,* [a]*jagt@illinois.edu*
[2]*Department of Biostatistics & Bioinformatics and Department of Computer Science, Duke University,* [b]*anru.zhang@duke.edu*

In this paper, we consider inference and uncertainty quantification for low Tucker rank tensors with additive noise in the high-dimensional regime. Focusing on the output of the *higher-order orthogonal iteration* (HOOI) algorithm, a commonly used algorithm for tensor singular value decomposition, we establish nonasymptotic distributional theory and study how to construct confidence regions and intervals for both the estimated singular vectors and the tensor entries in the presence of heteroskedastic subgaussian noise, which are further shown to be optimal for homoskedastic Gaussian noise. Furthermore, as a byproduct of our theoretical results, we establish the entrywise convergence of HOOI when initialized via diagonal deletion. To further illustrate the utility of our theoretical results, we then consider several concrete statistical inference tasks. First, in the tensor mixed-membership blockmodel, we consider a two-sample test for equality of membership profiles, and we propose a test statistic with consistency under local alternatives that exhibits a power improvement relative to the corresponding matrix test considered in several previous works. Next, we consider simultaneous inference for small collections of entries of the tensor, and we obtain consistent confidence regions. Finally, focusing on the particular case of testing whether entries of the tensor are equal, we propose a consistent test statistic that shows how index overlap results in different asymptotic standard deviations. All of our proposed procedures are fully data-driven, adaptive to noise distribution and signal strength, and do not rely on sample-splitting, and our main results highlight the effect of higher-order structures on estimation relative to the matrix setting. Our theoretical results are demonstrated through numerical simulations.

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LEARNING EXTREMAL GRAPHICAL STRUCTURES IN HIGH DIMENSIONS

BY SEBASTIAN ENGELKE[1,a], MICHAËL LALANCETTE[2,b] AND STANISLAV VOLGUSHEV[3,c]

[1]*Research Center for Statistics, University of Geneva,* [a]*sebastian.engelke@unige.ch*
[2]*Département de mathématiques, Université du Québec à Montréal,* [b]*lalancette.michael@uqam.ca*
[3]*Department of Statistical Sciences, University of Toronto,* [c]*stanislav.volgushev@utoronto.ca*

Extremal graphical models encode the conditional independence structure of multivariate extremes. Key statistics for learning extremal graphical structures are empirical extremal variograms, for which we prove nonasymptotic concentration bounds that hold under general domain of attraction conditions. For the popular class of Hüsler–Reiss models, we propose a majority voting algorithm for learning the underlying graph from data through L^1 regularized optimization. Our concentration bounds are used to derive explicit conditions that ensure the consistent recovery of any connected graph. The methodology is illustrated through a simulation study as well as the analysis of river discharge and currency exchange data.

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GRADIENT DESCENT INFERENCE IN EMPIRICAL RISK MINIMIZATION

BY QIYANG HAN[1,a] AND XIAOCONG XU[2,b]

[1]*Department of Statistics, Rutgers University,* [a]*qh85@stat.rutgers.edu*
[2]*Data Sciences and Operations Department, Marshall School of Business, University of Southern California,* [b]*xuxiaoco@marshall.usc.edu*

Gradient descent is one of the most widely used iterative algorithms in modern statistical learning. However, its precise algorithmic dynamics in high-dimensional settings remain only partially understood, which has limited its broader potential for statistical inference applications.

This paper provides a precise, nonasymptotic joint distributional characterization of gradient descent iterates and their debiased statistics in a broad class of empirical risk minimization problems, in the so-called mean-field regime where the sample size is proportional to the signal dimension. Our nonasymptotic state evolution theory holds for both general nonconvex loss functions and non-Gaussian data, and reveals the central role of two Onsager correction matrices that precisely characterize the nontrivial dependence among all gradient descent iterates in the mean-field regime.

Leveraging the joint state evolution characterization, we show that the gradient descent iterate retrieves approximate normality after a debiasing correction via a linear combination of observable loss derivative directions from all past iterates. Crucially, the debiasing coefficients are directly linked to the Onsager correction matrices, which can be estimated in a fully data-driven manner via the proposed *gradient descent inference algorithm*. This leads to a new algorithmic statistical inference framework based on debiased gradient descent, which (i) applies to a broad class of models with both convex and nonconvex losses, (ii) remains valid at each iteration without requiring algorithmic convergence and (iii) exhibits a certain robustness to possible model misspecification. As a by-product, our framework also provides algorithmic estimates of the generalization error at each iteration.

We demonstrate our theory and inference methods in the canonical single-index regression model and a generalized logistic regression model, where the natural loss functions may exhibit arbitrarily nonconvex landscapes. Our analysis further shows that, in linear regression with squared loss, the proposed debiased gradient descent iterate eventually coincides with the debiased convex regularized estimator in a mean-field distributional sense, and the quality of statistical inference for the unknown signal aligns exactly with the generalization error achieved along the algorithmic trajectory.

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STATISTICAL-COMPUTATIONAL TRADE-OFFS FOR RECURSIVE ADAPTIVE PARTITIONING ESTIMATORS

BY YAN SHUO TAN[1,a], JASON M. KLUSOWSKI[2,b] AND KRISHNAKUMAR BALASUBRAMANIAN[3,c]

[1]*Department of Statistics and Data Science, National University of Singapore,* [a]*yanshuo@nus.edu.sg*
[2]*Department of Operations Research and Financial Engineering, Princeton University,* [b]*jason.klusowski@princeton.edu*
[3]*Department of Statistics, University of California, Davis,* [c]*kbala@ucdavis.edu*

Models based on recursive adaptive partitioning such as decision trees and their ensembles are popular for high-dimensional regression as they can potentially avoid the curse of dimensionality. Because empirical risk minimization (ERM) is computationally infeasible, these models are typically trained using greedy algorithms. Although effective in many cases, these algorithms have been empirically observed to get stuck at local optima. We explore this phenomenon in the context of learning sparse regression functions over d binary features, showing that when the true regression function f^* does not satisfy Abbe et al. (In *Conference on Learning Theory* (2022) 4782–4887 PMLR)'s Merged Staircase Property (MSP), a form of heredity restriction similar to that used in classical ANOVA modeling, greedy training requires $\exp(\Omega(d))$ samples to achieve low estimation error. Conversely, when f^* does satisfy MSP, greedy training can attain small estimation error with only $O(\log d)$ samples. This dichotomy mirrors that of two-layer neural networks trained with stochastic gradient descent (SGD) in the mean-field regime, thereby establishing a head-to-head comparison between SGD-trained neural networks and greedy recursive partitioning estimators. Furthermore, ERM-trained recursive partitioning estimators achieve low estimation error with $O(\log d)$ samples irrespective of whether f^* satisfies MSP, thereby demonstrating a statistical-computational trade-off for greedy training. Our proofs are based on a novel interpretation of greedy recursive partitioning using stochastic process theory and a coupling technique that may be of independent interest.

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VECCHIA GAUSSIAN PROCESSES: ON PROBABILISTIC AND STATISTICAL PROPERTIES

BY BOTOND SZABO[1,a] AND YICHEN ZHU[2,b]

[1]*Bocconi Institute for Data Science and Analytics and Department of Decision Sciences, Bocconi University,* [a]*botond.szabo@unibocconi.it*

[2]*Department of Statistics and Actuarial Science, University of Hong Kong,* [b]*yczhu@hku.hk*

Gaussian Processes (GPs) are widely used to model dependencies in spatial statistics and machine learning. However, exact inference is computationally intractable for GP regression, with a time complexity of $O(n^3)$. The Vecchia approximation scales up computation by introducing sparsity into the spatial dependency structure, represented by a directed acyclic graph (DAG). Despite its practical popularity, this approach lacks rigorous theoretical foundations, and the choice of DAG structure remains an open problem.

In this paper, we systematically study the Vecchia approximation of the popular, isotropic Matérn GP as standalone stochastic process and uncover key probabilistic and statistical properties. We propose selecting parent sets as norming sets with fixed cardinality in the Vecchia approximation. On the probabilistic side, we show that the conditional distributions of Matérn GPs, as well as their Vecchia approximations, can be characterized by polynomial interpolations. This enables us to establish several results on small ball probabilities and the Reproducing Kernel Hilbert Spaces (RKHSs) of Vecchia GPs. Building on these probabilistic results, we prove that in the nonparametric regression model, the corresponding posterior contracts around the truth at the optimal minimax rate, both under oracle rescaling and hierarchical tuning of the prior.

We illustrate the theoretical findings through numerical experiments on synthetic datasets. Our core algorithms are implemented in C++ with an R interface.

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UNCERTAINTY QUANTIFICATION FOR ITERATIVE ALGORITHMS IN LINEAR MODELS WITH APPLICATION TO EARLY STOPPING

BY PIERRE C. BELLEC[1,a] AND KAI TAN[2,b]

[1]*Department of Statistics, Rutgers University,* [a]*pierre.bellec@rutgers.edu*
[2]*Department of Statistics, The Ohio State University,* [b]*tan.1457@osu.edu*

This paper investigates the iterates $\widehat{\boldsymbol{b}}^1, \ldots, \widehat{\boldsymbol{b}}^T$ obtained from iterative algorithms in high-dimensional linear regression problems, in the regime where the feature dimension p is comparable with the sample size n, that is, $p \asymp n$. The analysis and proposed estimators are applicable to Gradient Descent (GD), proximal GD and their accelerated variants such as Fast Iterative Soft-Thresholding (FISTA). The paper proposes novel estimators for the generalization error of the iterate $\widehat{\boldsymbol{b}}^t$ for any fixed iteration t along the trajectory. These estimators are proved to be $\sqrt{n}$-consistent under Gaussian designs. Applications to early-stopping are provided: when the generalization error of the iterates is a U-shape function of the iteration t, the estimates allow to select from the data an iteration $\hat{t}$ that achieves the smallest generalization error along the trajectory. Additionally, we provide a technique for developing debiasing corrections and valid confidence intervals for the components of the true coefficient vector from the iterate $\widehat{\boldsymbol{b}}^t$ at any finite iteration t. Extensive simulations on synthetic data illustrate the theoretical results.

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MARKOV STICK-BREAKING PROCESSES

BY MARÍA F. GIL-LEYVA[1,a], ANTONIO LIJOI[2,c], RAMSÉS H. MENA[1,b] AND IGOR PRÜNSTER[2,d]

[1]*Department of Probability and Statistics, IIMAS, UNAM,* [a]*marifer@sigma.iimas.unam.mx,* [b]*ramses@sigma.iimas.unam.mx*
[2]*Bocconi Institute for Data Science and Analytics, Bocconi University,* [c]*antonio.lijoi@unibocconi.it,* [d]*igor@unibocconi.it*

Stick-breaking has a long history and is one of the most popular procedures for constructing random discrete distributions in statistics and machine learning. In particular, due to their intuitive construction and computational tractability they are ubiquitous in modern Bayesian nonparametric inference. Most widely used models, such as the Dirichlet and the Pitman–Yor processes, rely on i.i.d. or independent length variables. Here, we pursue a completely unexplored research direction by considering Markov length variables and investigate the corresponding general class of stick-breaking processes, which we term *Markov stick-breaking processes*. We establish conditions under which the associated species sampling process is proper and the distribution of a Markov stick-breaking process has full topological support, two fundamental *desiderata* for Bayesian nonparametric models. We also analyze the stochastic ordering of the weights and provide a new characterization of the Pitman–Yor process as the only stick-breaking process invariant under size-biased permutations, under mild conditions. Moreover, we identify two notable subclasses of Markov stick-breaking processes that enjoy appealing properties and include Dirichlet, Pitman–Yor and geometric priors as special cases. Our findings include distributional results enabling posterior inference algorithms and methodological insights.

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IDENTIFICATION AND ESTIMATION FOR MATRIX TIME-SERIES CP-FACTOR MODELS

BY JINYUAN CHANG[1,2,3,a], YUE DU[1,b], GUANGLIN HUANG[1,c] AND QIWEI YAO[4,d]

[1]*Joint Laboratory of Data Science and Business Intelligence, Southwestern University of Finance and Economics,* [a]*changjinyuan@swufe.edu.cn,* [b]*122071400004@smail.swufe.edu.cn,* [c]*huanggl@swufe.edu.cn*

[2]*State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences*

[3]*China Center for Economic Research, Peking University*

[4]*Department of Statistics, London School of Economics and Political Science,* [d]*q.yao@lse.ac.uk*

We propose a new method for identifying and estimating the CP-factor models for matrix time series. Unlike the generalized eigenanalysis-based method (*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **85** (2023) 127–148) for which the convergence rates of the associated estimators may suffer from small eigengaps as the asymptotic theory is based on some matrix perturbation analysis, the proposed new method enjoys faster convergence rates which are free from any eigengaps. It achieves this by turning the problem into a joint diagonalization of several matrices whose elements are determined by a basis of a linear system, and by choosing the basis carefully to avoid near colinearity (see Proposition 5 and Section 4.3). Furthermore, unlike the generalized eigenanalysis-based method which requires the two factor loading matrices to be full-ranked, the proposed new method can handle rank-deficient factor loading matrices. Illustration with both simulated and real matrix time-series data shows the advantages of the proposed new method.

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TOWARDS A UNIFIED THEORY FOR SEMIPARAMETRIC DATA FUSION WITH INDIVIDUAL-LEVEL DATA

BY ELLEN GRAHAM[a], MARCO CARONE[b] AND ANDREA ROTNITZKY[c]

Department of Biostatistics, University of Washington, [a]*graham18@uw.edu,* [b]*mcarone@uw.edu,* [c]*arotnitz@uw.edu*

We consider inference about a finite-dimensional parameter integrating samples from independent sources. A recently developed theory considers scenarios where sources align with subsets of the conditional distributions of a single factorization of the joint target distribution. While this theory applies in many settings, it falls short in important data fusion problems, such as two-sample instrumental variable analysis, settings that integrate data from epidemiological studies with diverse designs, and studies with mismeasured variables supplemented by external validation studies. In this paper we derive a comprehensive theory that, in particular, covers these settings by allowing the integration of sources aligned with conditional distributions that do not correspond to a single factorization of the target distribution. We provide a universal characterization of the influence functions of regular and asymptotically linear estimators and the efficient influence function of a target parameter, irrespective of the parameter of interest or the statistical model for the target distribution, thus paving the way for a unified theory for machine-learning debiased, semiparametric efficient estimation.

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A TWO-STEP ESTIMATING APPROACH FOR HEAVY-TAILED AR MODELS WITH NONZERO MEDIAN GARCH-TYPE NOISES

BY RUI SHE[1,a], LINLIN DAI[1,b] AND SHIQING LING[2,c]

[1]*Center of Statistical Research, School of Statistics and Data Science, The Southwestern University of Finance and Economics,* [a]*rshe@swufe.edu.cn,* [b]*ldaiab@swufe.edu.cn*

[2]*Department of Mathematics, The Hong Kong University of Science and Technology,* [c]*maling@ust.hk*

This paper develops a novel two-step estimating procedure for heavy-tailed AR models with nonzero median GARCH-type noises, allowing for time-varying volatility. We first establish the self-weighted quantile regression estimator (SQE) across all quantile levels $\tau \in (0, 1)$ for the AR parameters θ_0. We show that the SQE, less a bias, converges weakly to a Gaussian process at a rate of $n^{-1/2}$. The bias is zero if and only if τ equals τ_0, the probability that the noise is less than zero. Based on the SQE, we propose an approach to estimate τ_0 in the second step and feed the estimated τ_0 back into the SQE to estimate θ_0. Both the estimated τ_0 and θ_0 are shown to be consistent and asymptotically normal. A random weighting bootstrap method is developed to approximate the complex distribution. The problem we study is nonstandard because τ_0 may not be identifiable in conventional quantile regression, and the usual methods cannot verify the existence of the SQE bias. Unlike existing procedures for heavy-tailed time series, our method does not require prior information about the symmetry, tail index, or the parametric form of the noise, nor does it require classical identification conditions, such as zero-mean or zero-median.

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TEST OF INDEPENDENCE USING GENERALIZED DISTANCE CORRELATION

BY JIANQING FAN[1,a], ZHIPENG LOU[2,b] AND DANNA ZHANG[2,c]

[1]*Department of Operations Research and Financial Engineering, Princeton University,* [a]*jqfan@princeton.edu*
[2]*Department of Mathematics, University of California, San Diego,* [b]*zlou@ucsd.edu,* [c]*daz076@ucsd.edu*

We study the fundamental statistical inference concerning the testing of independence between two random vectors. Existing asymptotic theories for test statistics based on distance covariance can only apply to either low-dimensional or high-dimensional settings and require stringent distributional assumptions. In this work we develop a new unified distributional theory of the sample generalized distance covariance that works for random vectors of arbitrary dimensions under fairly mild moment conditions. In particular, a Gaussian approximation result is established with a nonasymptotic error bound, and the asymptotic null distribution of the sample generalized distance covariance is shown to be distributed as a linear combination of independently and identically distributed chi-squared random variables. To estimate the asymptotic null distribution practically, we propose a half-permutation procedure and provide the theoretical justification for its validity. The exact asymptotic distribution of the resampling distribution is derived under general marginal moment conditions, and the proposed procedure is shown to be asymptotically equivalent to the oracle procedure with known marginal distributions.

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ATTAINABILITY OF TWO-POINT TESTING RATES FOR FINITE-SAMPLE LOCATION ESTIMATION

BY SPENCER COMPTON[a] AND GREGORY VALIANT[b]

Department of Computer Science, Stanford University, [a]*comptons@stanford.edu,* [b]*valiant@stanford.edu*

Le Cam's two-point testing method yields perhaps the simplest lower bound for estimating the mean of a distribution: roughly, if it is impossible to well distinguish a distribution centered at μ from the same distribution centered at $\mu + \Delta$, then it is impossible to estimate the mean by better than $\Delta/2$. It is setting-dependent, whether or not a nearly matching upper bound is attainable. We study the conditions under which the two-point testing lower bound can be attained for univariate mean estimation; both in the setting of *location estimation* (where the distribution is known up to translation) and *adaptive location estimation* (unknown distribution). Roughly, we will say an estimate nearly attains the two-point testing lower bound if it incurs error that is at most polylogarithmically larger than the *Hellinger modulus of continuity* for $\tilde{\Omega}(n)$ samples.

Adaptive location estimation is particularly interesting, as some distributions admit much better guarantees than sub-Gaussian rates (e.g., $\mathrm{Unif}(\mu - 1, \mu + 1)$ permit error $\Theta(\frac{1}{n})$, while the sub-Gaussian rate is $\Theta(\frac{1}{\sqrt{n}})$), yet it is not obvious whether these rates may be adaptively attained by one unified approach. Our main result designs an algorithm that nearly attains the two-point testing rate for mixtures of symmetric, log-concave distributions with a common mean. Moreover, this algorithm runs in near-linear time and is parameter-free. In contrast, we show the two-point testing rate is not nearly attainable, even for symmetric, unimodal distributions.

We complement this with results for location estimation, showing the two-point testing rate is nearly attainable for unimodal distributions but unattainable for symmetric distributions.

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MINIMAX OPTIMAL SERIATION IN POLYNOMIAL TIME

BY YANN ISSARTEL[1,a], CHRISTOPHE GIRAUD[2,b] AND NICOLAS VERZELEN[3,c]

[1]*LTCI, Télécom Paris, Institut Polytechnique de Paris,* [a]*yann.issartel@telecom-paris.fr*
[2]*Laboratoire de Mathématiques d'Orsay, Université Paris-Saclay,* [b]*christophe.giraud@universite-paris-saclay.fr*
[3]*MISTEA, INRAE, Institut Agro, Université Montpellier,* [c]*Nicolas.Verzelen@inrae.fr*

We consider the seriation problem, whose goal is to recover a hidden ordering from a noisy observation of a permuted Robinson matrix. We establish sharp minimax rates under average-Lipschitz conditions that strictly extend the bi-Lipschitz framework of Giraud, Issartel and Verzelen (*Electron. J. Stat.* (2023) **17** 1587–1662). We further design a polynomial-time algorithm that attains these optimal rates, thereby resolving two open questions raised in Giraud, Issartel and Verzelen. Finally, our analysis extends to a broader class of matrices beyond those generated by exact permutations.

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DUAL INDUCTION CLT FOR HIGH-DIMENSIONAL *m*-DEPENDENT DATA

BY HEEJONG BONG[1,a], ARUN KUMAR KUCHIBHOTLA[2,b] AND ALESSANDRO RINALDO[3,c]

[1]*Department of Statistics, Purdue University,* [a]*bong0@purdue.edu*
[2]*Department of Statistics and Data Science, Carnegie Mellon University,* [b]*arunku@stat.cmu.edu*
[3]*Department of Statistics and Data Science, University of Texas at Austin,* [c]*alessandro.rinaldo@austin.utexas.edu*

We derive novel and sharp high-dimensional Berry–Esseen bounds for the sum of m-dependent random vectors over the class of hyperrectangles exhibiting only a poly-logarithmic dependence in the dimension. Our results hold under minimal assumptions, such as nondegenerate covariances and finite third moments, and exhibit an optimal sample complexity of order $m^{(q-1)/(q-2)}/\sqrt{n}$. Aside from logarithmic terms, the resulting rates match the optimal rates established in the univariate case. When specialized to the sums of independent nondegenerate random vectors, our results produce sharp and, in some cases, optimal rates under the weakest possible conditions. We develop a novel inductive relationship between anticoncentration inequalities and Berry–Esseen bounds inspired by the classical Lindeberg swapping method and the concentration inequality approach for dependent data that may be of independent interest.

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STATISTICAL INFERENCE IN TENSOR COMPLETION: OPTIMAL UNCERTAINTY QUANTIFICATION AND STATISTICAL-TO-COMPUTATIONAL GAPS

BY WANTENG MA[1,a] AND DONG XIA[2,b]

[1]*Department of Statistics and Data Science, University of Pennsylvania,* [a]*wanteng@wharton.upenn.edu*
[2]*Department of Mathematics, Hong Kong University of Science and Technology,* [b]*madxia@ust.hk*

This paper presents a simple yet efficient method for statistical inference of tensor linear forms using incomplete and noisy observations. Under the Tucker low-rank tensor model and the missing-at-random assumption, we utilize an appropriate initial estimate along with a debiasing technique followed by a one-step power iteration to construct an asymptotically normal test statistic. This method is suitable for various statistical inference tasks, including constructing confidence intervals, inference under heteroskedastic and sub-exponential noise, and simultaneous testing. We demonstrate that the estimator achieves the Cramér–Rao lower bound on Riemannian manifolds, indicating its optimality in uncertainty quantification. We comprehensively examine statistical-to-computational gaps and investigate the impact of initialization on the minimal conditions regarding sample sizes and signal-to-noise ratios required for accurate inference. Our findings show that, with independent initialization, statistically optimal sample sizes and signal-to-noise ratios are sufficient for accurate inference. Conversely, if only dependent initialization is available, computationally optimal sample sizes and signal-to-noise ratios still guarantee asymptotic normality without the need for data-splitting. We present the phase transition between computational and statistical limits. Numerical simulation results align with the theoretical findings.

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A NOVEL STATISTICAL APPROACH TO ANALYZE IMAGE CLASSIFICATION

BY JUNTONG CHEN[1,a], SOPHIE LANGER[2,b] AND JOHANNES SCHMIDT-HIEBER[3,c]

[1]*School of Mathematical Sciences, Xiamen University,* [a]*juntong.chen@xmu.edu.cn*
[2]*Fakultät für Mathematik, Ruhr-Universität Bochum,* [b]*s.langer@rub.de*
[3]*Department of Applied Mathematics, University of Twente,* [c]*a.j.schmidt-hieber@utwente.nl*

The recent statistical theory of neural networks focuses on nonparametric denoising problems that treat randomness as additive noise. Variability in image classification datasets does, however, not originate from additive noise but from variation of the shape and other characteristics of the same object across different images. To address this problem, we introduce a tractable model for supervised image classification. While from the function estimation point of view, every pixel in an image is a variable, and large images lead to high-dimensional function recovery tasks suffering from the curse of dimensionality, increasing the number of pixels in the proposed image deformation model enhances the image resolution and makes the object classification problem easier. We introduce and theoretically analyze three approaches. Two methods combine image alignment with a one-nearest neighbor classifier. Under a separation condition, it is shown that perfect classification is possible. The third method fits a convolutional neural network (CNN) to the data. We derive a rate for the misclassification error that depends on the sample size and the complexity of the deformation class. An empirical study corroborates the theoretical findings.

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APPROXIMATE INDEPENDENCE OF PERMUTATION MIXTURES

BY YANJUN HAN[a] AND JONATHAN NILES-WEED[b]

Courant Institute of Mathematical Sciences and Center for Data Science, New York University, [a]*yanjunhan@nyu.edu,* [b]*jnw@cims.nyu.edu*

We prove bounds on statistical distances between high-dimensional exchangeable mixture distributions (which we call *permutation mixtures*) and their i.i.d. counterparts. Our results are based on a novel method for controlling χ^2 divergences between exchangeable mixtures, which is tighter than the existing methods of moments or cumulants. At a technical level, a key innovation in our proofs is a new Maclaurin-type inequality for elementary symmetric polynomials of variables that sum to zero and an upper bound on permanents of doubly-stochastic positive semidefinite matrices. We obtain as a corollary a new de Finetti-style theorem (in the language of Diaconis and Freedman, 1987) as well as several new statistical results, including a differential privacy guarantee for the "shuffled privacy model" with Gaussian noise and improved generic consistency guarantees for empirical Bayes procedures in compound decision problems.

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QUASI-MONTE CARLO CONFIDENCE INTERVALS USING QUANTILES OF RANDOMIZED NETS

BY ZEXIN PAN[a]

Institute of Fundamental and Transdisciplinary Research, Zhejiang University, [a]*zep002@zju.edu.cn*

Recent advances in quasi-Monte Carlo integration have shown that for linearly scrambled digital net estimators, the convergence rate can be dramatically improved by taking the median rather than the mean of multiple independent replicates. In this work, we demonstrate that the quantiles of such estimators can be used to construct confidence intervals with asymptotically valid coverage for high-dimensional integrals. By analyzing the error distribution for a class of infinitely differentiable integrands, we prove that as the sample size increases, the integration error decomposes into an asymptotically symmetric component and a vanishing remainder. Consequently, the asymptotic error distribution is symmetric about zero, ensuring that a quantile-based interval constructed from independent replicates captures the true integral with probability converging to a nominal level determined by the binomial distribution.

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ERRATUM: QUANTILE PROCESSES AND THEIR APPLICATIONS IN FINITE POPULATIONS

BY ANURAG DEY[a] AND PROBAL CHAUDHURI[b]

Statistics and Mathematics unit, Indian Statistical Institute, [a]*deyanuragsaltlake64@gmail.com,* [b]*probalchaudhuri@gmail.com*

This erratum contains some corrections to Dey and Chaudhuri (*Ann. Statist.* **52** (2024) 2194–2216).

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ERRATUM: EDGEWORTH EXPANSIONS FOR LINEAR RANK STATISTICS

BY WALTER SCHNELLER[a]

Fakultät Angewandte Natur- und Geisteswissenschaften, Technische Hochschule Würzburg-Schweinfurt,
[a]*walter.schneller@thws.de*

This erratum contains a correction of the incorrect defined distribution on page 1109, last two lines, in Schneller (*Ann. Statist.* **17** (1989) 1103–1123). Additionally, some typographical errors are corrected.

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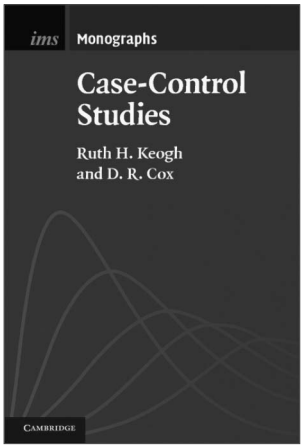

Case-Control Studies

Ruth H. Keogh
and D. R. Cox

The case-control approach is a powerful method for investigating factors that may explain a particular event. It is extensively used in epidemiology to study disease incidence, one of the best-known examples being Bradford Hill and Doll's investigation of the possible connection between cigarette smoking and lung cancer. More recently, case-control studies have been increasingly used in other fields, including sociology and econometrics.

With a particular focus on statistical analysis, this book is ideal for applied and theoretical statisticians wanting an up-to-date introduction to the field. It covers the fundamentals of case-control study design and analysis as well as more recent developments, including two-stage studies, case-only studies and methods for case-control sampling in time. The latter have important applications in large prospective cohorts which require case-control sampling designs to make efficient use of resources. More theoretical background is provided in an appendix for those new to the field.

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