# BERNOULLI

*Official Journal of the Bernoulli Society for Mathematical Statistics and Probability*

Volume Twenty Five Number Three August 2019 ISSN: 1350-7265

CONTENTS



*(continued)*

The papers published in Bernoulli are indexed or abstracted in *Current Index to Statistics*, *Mathematical Reviews*, *Statistical Theory and Method Abstracts-Zentralblatt (STMA-Z)*, and *Zentralblatt für Mathematik* (also avalaible on the *MATH via STN* database and *Compact MATH CD-ROM*). A list of forthcoming papers can be found online at http://www.bernoulli-society.org/index.php/publications/bernoulli-journal/bernoulli-journal-papers

# BERNOULLI

*Official Journal of the Bernoulli Society for Mathematical Statistics and Probability*

Volume Twenty Five Number Three August 2019 ISSN: 1350-7265

CONTENTS

*(continued)*

BERNOULLI Volume 25 Number 3 August 2019 Pages 1603–2408 ISI/BS

Volume 25 Number 3    August 2019    ISSN 1350-7265

# BERNOULLI

**Official Journal of the Bernoulli Society for Mathematical Statistics and Probability**

**Aims and Scope**

**BERNOULLI** is the journal of the Bernoulli Society for Mathematical Statistics and Probability, issued four times per year. The journal provides a comprehensive account of important developments in the fields of statistics and probability, offering an international forum for both theoretical and applied work.

**Bernoulli Society for Mathematical Statistics and Probability**

The Bernoulli Society was founded in 1973. It is an autonomous Association of the International Statistical Institute, ISI. According to its statutes, the object of the Bernoulli Society is the advancement, through international contacts, of the sciences of probability (including the theory of stochastic processes) and mathematical statistics and of their applications to all those aspects of human endeavour which are directed towards the increase of natural knowledge and the welfare of mankind.

**Meetings: http://www.bernoulli-society.org/index.php/meetings**

The Society holds a World Congress every four years; more frequent meetings, coordinated by the Society's standing committees and often organised in collaboration with other organisations, are the European Meeting of Statisticians, the Conference on Stochastic Processes and their Applications, the CLAPEM meeting (Latin-American Congress on Probability and Mathematical Statistics), the European Young Statisticians Meeting, and various meetings on special topics – in the physical sciences in particular. The Society, as an association of the ISI, also collaborates with other ISI associations in the organization of the biennial ISI World Statistics Congresses (formerly ISI Sessions).

**Executive Committee**



---

The papers published in Bernoulli are indexed or abstracted in *Current Index to Statistics, Mathematical Reviews, Statistical Theory and Method Abstracts-Zentralblatt (STMA-Z), Thomson Scientific* and *Zentralblatt für Mathematik* (also available on the *MATH via STN* database and *Compact MATH CD-ROM*). A list of forthcoming papers can be found online at http://www.bernoulli-society.org/index.php/publications/bernoulli-journal/bernoulli-journal-papers



In 2019 Bernoulli consists of 4 issues published in February, May, August and November.

Bernoulli Society
for Mathematical Statistics
and Probability

*Bernoulli* **25**(3), 2019, 1603–1639
https://doi.org/10.3150/17-BEJ978

# Sparse Hanson–Wright inequalities for subgaussian quadratic forms

SHUHENG ZHOU

*Department of Statistics, University of California, Riverside, CA 92521, USA. E-mail: szhou@ucr.edu*

In this paper, we provide a proof for the Hanson–Wright inequalities for sparse quadratic forms in subgaussian random variables. This provides useful concentration inequalities for sparse subgaussian random vectors in two ways. Let $X = (X_1, \dots, X_m) \in \mathbf{R}^m$ be a random vector with independent subgaussian components, and $\xi = (\xi_1, \dots, \xi_m) \in \{0, 1\}^m$ be independent Bernoulli random variables. We prove the large deviation bound for a sparse quadratic form of $(X \circ \xi)^T A(X \circ \xi)$, where $A \in \mathbf{R}^{m\times m}$ is an $m \times m$ matrix, and random vector $X \circ \xi$ denotes the Hadamard product of an isotropic subgaussian random vector $X \in \mathbf{R}^m$ and a random vector $\xi \in \{0, 1\}^m$ such that $(X \circ \xi)_i = X_i\xi_i$, where $\xi_1, \dots, \xi_m$ are independent Bernoulli random variables. The second type of sparsity in a quadratic form comes from the setting where we randomly sample the elements of an anisotropic subgaussian vector $Y = HX$ where $H \in \mathbf{R}^{m\times m}$ is an $m \times m$ symmetric matrix; we study the large deviation bound on the $\ell_2$-norm $\|D_\xi Y\|_2^2$ from its expected value, where for a given vector $x \in \mathbf{R}^m$, $D_x = \mathrm{diag}(x)$ denotes the diagonal matrix whose main diagonal entries are the entries of $x$. This form arises naturally from the context of covariance estimation.

*Keywords:* Hanson–Wright inequality; sparse quadratic forms; subgaussian concentration

## References

[1] Adamczak, R. and Wolff, P. (2015). Concentration inequalities for non-Lipschitz functions with bounded derivatives of higher order. *Probab. Theory Related Fields* **162** 531–586. MR3383337
[2] Barthe, F. and Milman, E. (2012). Transference principles for log-Sobolev and spectral-gap with applications to conservative spin systems.
[3] Dawid, A.P. (1981). Some matrix-variate distribution theory: Notational considerations and a Bayesian application. *Biometrika* **68** 265–274. MR0614963
[4] de la Peña, V.H. and Giné, E. (1999). *Decoupling: From Dependence to Independence: Randomly Stopped Processes. U-Statistics and Processes. Martingales and Beyond. Probability and Its Applications (New York)*. New York: Springer. MR1666908
[5] de la Peña, V.H. and Montgomery-Smith, S.J. (1995). Decoupling inequalities for the tail probabilities of multivariate $U$-statistics. *Ann. Probab.* **23** 806–816. MR1334173
[6] Diakonikolas, I., Kane, D.M. and Nelson, J. (2010). Bounded independence fools degree-2 threshold functions. In 2010 *IEEE* 51*st Annual Symposium on Foundations of Computer Science—FOCS* 2010 11–20. Los Alamitos, CA: IEEE Computer Soc. MR3024771
[7] Foucart, S. and Rauhut, H. (2013). *A Mathematical Introduction to Compressive Sensing. Applied and Numerical Harmonic Analysis*. New York: Birkhäuser/Springer. MR3100033
[8] Gupta, A.K. and Varga, T. (1992). Characterization of matrix variate normal distributions. *J. Multivariate Anal.* **41** 80–88. MR1156682
[9] Hanson, D.L. and Wright, F.T. (1971). A bound on tail probabilities for quadratic forms in independent random variables. *Ann. Math. Stat.* **42** 1079–1083. MR0279864

1350-7265 

[10] Horn, R.A. and Johnson, C.R. (1991). *Topics in Matrix Analysis*. Cambridge: Cambridge Univ. Press. MR1091716

[11] Hsu, D., Kakade, S.M. and Zhang, T. (2012). Tail inequalities for sums of random matrices that depend on the intrinsic dimension. *Electron*. *Commun*. *Probab.* **17** Art. ID 14. MR2900355

[12] Latała, R. (2006). Estimates of moments and tails of Gaussian chaoses. *Ann*. *Probab.* **34** 2315–2331. MR2294983

[13] Rudelson, M. (2016). On the complexity of the set of unconditional convex bodies. *Discrete Comput*. *Geom*. **55** 185–202. MR3439264

[14] Rudelson, M. and Vershynin, R. (2013). Hanson–Wright inequality and sub-Gaussian concentration. *Electron*. *Commun*. *Probab.* **18** Art. ID 82. MR3125258

[15] Talagrand, M. (1995). Sections of smooth convex bodies via majorizing measures. *Acta Math*. **175** 273–300. MR1368249

[16] Vershynin, R. (2011). A simple decoupling inequality in probability theory. Available at http://www-personal.umich.edu/~romanv/papers/papers.html.

[17] Vershynin, R. (2012). Introduction to the non-asymptotic analysis of random matrices. In *Compressed Sensing* 210–268. Cambridge: Cambridge Univ. Press. MR2963170

[18] Wright, F.T. (1973). A bound on tail probabilities for quadratic forms in independent random variables whose distributions are not necessarily symmetric. *Ann*. *Probab.* **1** 1068–1070. MR0353419

[19] Zhou, S. (2014). Gemini: Graph estimation with matrix variate normal instances. *Ann*. *Statist*. **42** 532–562.

[20] Zhou, S. (2014). Supplement to "Gemini: Graph estimation with matrix variate normal instances". *Ann*. *Statist*. DOI:10.1214/13-AOS1187SUPP.

[21] Zhou, S. (2019). The concentration of measure phenomenon on sparse matrix variate random matrices. Working paper.

*Bernoulli* **25**(3), 2019, 1640–1658
https://doi.org/10.3150/17-BEJ1015

# Dual attainment for the martingale transport problem

MATHIAS BEIGLBÖCK[1], TONGSEOK LIM[2] and JAN OBŁÓJ[3]

[1] *Faculty of Mathematics, University of Vienna, Oskar Morgensternplatz 1, 6 floor, 1090 Wien, Austria. E-mail: mathias.beiglboeck@univie.ac.at*
[2] *Institute of Mathematical Sciences, ShanghaiTech University, 393 Middle Huaxia Road, Pudong, Shanghai, 201210, China. E-mail: tlim@shanghaitech.edu.cn*
[3] *Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, UK. E-mail: jan.obloj@maths.ox.ac.uk*

We investigate existence of dual optimizers in one-dimensional martingale optimal transport problems. While [*Ann. Probab.* **45** (2017) 3038–3074] established such existence for weak (quasi-sure) duality, [*Finance Stoch.* **17** (2013) 477–501] showed existence for the natural stronger (pointwise) duality may fail even in regular cases. We establish that (pointwise) dual maximizers exist when $y \mapsto c(x, y)$ is convex, or equivalent to a convex function. It follows that when marginals are compactly supported, the existence holds when the cost $c(x, y)$ is twice continuously differentiable in $y$. Further, this may not be improved as we give examples with $c(x, \cdot) \in C^{2-\varepsilon}$, $\varepsilon > 0$, where dual attainment fails. Finally, when measures are compactly supported, we show that dual optimizers are Lipschitz if $c$ is Lipschitz.

*Keywords:* dual attainment; Kantorovich duality; martingale optimal transport; robust mathematical finance

## References

[1] Beiglböck, M., Cox, A.M.G. and Huesmann, M. (2017). Optimal transport and Skorokhod embedding. *Invent. Math.* **208** 327–400. MR3639595
[2] Beiglböck, M., Henry-Labordère, P. and Penkner, F. (2013). Model-independent bounds for option prices—A mass transport approach. *Finance Stoch.* **17** 477–501. MR3066985
[3] Beiglböck, M. and Juillet, N. (2016). On a problem of optimal transport under marginal martingale constraints. *Ann. Probab.* **44** 42–106. MR3456332
[4] Beiglböck, M. and Nutz, M. (2014). Martingale inequalities and deterministic counterparts. *Electron. J. Probab.* **19** no. 95, 15. MR3272328
[5] Beiglböck, M., Nutz, M. and Touzi, N. (2017). Complete duality for martingale optimal transport on the line. *Ann. Probab.* **45** 3038–3074. MR3706738
[6] Beiglböck, M. and Pratelli, A. (2012). Duality for rectified cost functions. *Calc. Var. Partial Differential Equations* **45** 27–41. MR2957649
[7] Campi, L., Laachir, I. and Martini, C. (2017). Change of numeraire in the two-marginals martingale transport problem. *Finance Stoch.* **21** 471–486. MR3626622
[8] Dolinsky, Y. and Soner, H.M. (2014). Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Related Fields* **160** 391–427. MR3256817
[9] Galichon, A., Henry-Labordère, P. and Touzi, N. (2014). A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.* **24** 312–336. MR3161649

1350-7265 

[10] Ghoussoub, N., Kim, Y.-H. and Lim, T. (2019). Structure of optimal martingale transport plans in general dimensions. *Ann. Probab.* **47** 109–164. MR3909967

[11] Henry-Labordère, P., Obłój, J., Spoida, P. and Touzi, N. (2016). The maximum maximum of a martingale with given $n$ marginals. *Ann. Appl. Probab.* **26** 1–44. MR3449312

[12] Hobson, D. (2011). The Skorokhod embedding problem and model-independent bounds for option prices. In *Paris-Princeton Lectures on Mathematical Finance* 2010. *Lecture Notes in Math.* **2003** 267–318. Berlin: Springer. MR2762363

[13] Hobson, D. and Klimmek, M. (2015). Robust price bounds for the forward starting straddle. *Finance Stoch.* **19** 189–214. MR3292129

[14] Hobson, D. and Neuberger, A. (2012). Robust bounds for forward start options. *Math. Finance* **22** 31–56. MR2881879

[15] Jacod, J. and Mémin, J. (1981). Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité. In *Séminaire de Probabilités XV*. *Lecture Notes in Math.* **850** 529–546. Cham: Springer. MR0622586

[16] Nutz, M., Stebegg, F. and Tan, X. (2017). Multiperiod martingale transport. ArXiv e-prints.

[17] Obłój, J. (2004). The Skorokhod embedding problem and its offspring. *Probab. Surv.* **1** 321–390. MR2068476

[18] Obłój, J., Spoida, P. and Touzi, N. (2015). Martingale inequalities for the maximum via pathwise arguments. In *In Memoriam Marc Yor—Séminaire de Probabilités XLVII*. *Lecture Notes in Math.* **2137** 227–247. Cham: Springer. MR3444301

[19] Strassen, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Stat.* **36** 423–439. MR0177430

*Bernoulli* **25**(3), 2019, 1659–1689
https://doi.org/10.3150/18-BEJ1031

# Martingale decompositions and weak differential subordination in UMD Banach spaces

IVAN S. YAROSLAVTSEV

*Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands. E-mail: I.S.Yaroslavtsev@tudelft.nl*

In this paper, we consider Meyer–Yoeurp decompositions for UMD Banach space-valued martingales. Namely, we prove that $X$ is a UMD Banach space if and only if for any fixed $p \in (1, \infty)$, any $X$-valued $L^p$-martingale $M$ has a unique decomposition $M = M^d + M^c$ such that $M^d$ is a purely discontinuous martingale, $M^c$ is a continuous martingale, $M_0^c = 0$ and

$$\mathbb{E}\|M_\infty^d\|^p + \mathbb{E}\|M_\infty^c\|^p \le c_{p,X}\mathbb{E}\|M_\infty\|^p.$$

An analogous assertion is shown for the Yoeurp decomposition of a purely discontinuous martingales into a sum of a quasi-left continuous martingale and a martingale with accessible jumps.

As an application, we show that $X$ is a UMD Banach space if and only if for any fixed $p \in (1, \infty)$ and for all $X$-valued martingales $M$ and $N$ such that $N$ is weakly differentially subordinated to $M$, one has the estimate $\mathbb{E}\|N_\infty\|^p \le C_{p,X}\mathbb{E}\|M_\infty\|^p$.

*Keywords:* accessible jumps; Brownian representation; Burkholder function; canonical decomposition of martingales; continuous martingales; differential subordination; Meyer–Yoeurp decomposition; purely discontinuous martingales; quasi-left continuous; stochastic integration; UMD Banach spaces; weak differential subordination; Yoeurp decomposition

## References

[1] Albeverio, S. and Rüdiger, B. (2005). Stochastic integrals and the Lévy–Ito decomposition theorem on separable Banach spaces. *Stoch. Anal. Appl.* **23** 217–253. MR2130348

[2] Applebaum, D. (2007). Lévy processes and stochastic integrals in Banach spaces. *Probab. Math. Statist.* **27** 75–88. MR2353272

[3] Bañuelos, R., Bielaszewski, A. and Bogdan, K. (2011). Fourier multipliers for non-symmetric Lévy processes. In *Marcinkiewicz Centenary Volume. Banach Center Publ.* **95** 9–25. Warsaw: Polish Acad. Sci. Inst. Math. MR2918086

[4] Bañuelos, R. and Bogdan, K. (2007). Lévy processes and Fourier multipliers. *J. Funct. Anal.* **250** 197–213. MR2345912

[5] Baumgartner, F. (2015). Lévy processes with values in locally convex Suslin spaces. Available at arXiv:1510.00538.

[6] Bourgain, J. (1983). Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.* **21** 163–168. MR0727340

[7] Brzeźniak, Z. (1995). Stochastic partial differential equations in M-type 2 Banach spaces. *Potential Anal.* **4** 1–45. MR1313905
[8] Burkholder, D.L. (1983). A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In *Conference on Harmonic Analysis in Honor of Antoni Zygmund*, *Vol. I*, *II* (*Chicago*, *Ill.*, 1981). *Wadsworth Math. Ser.* 270–286. Belmont, CA: Wadsworth. MR0730072
[9] Burkholder, D.L. (1986). Martingales and Fourier analysis in Banach spaces. In *Probability and Analysis* (*Varenna*, 1985). *Lecture Notes in Math.* **1206** 61–108. Berlin: Springer. MR0864712
[10] Burkholder, D.L. (2001). Martingales and singular integrals in Banach spaces. In *Handbook of the Geometry of Banach Spaces*, *Vol. I* 233–269. Amsterdam: North-Holland. MR1863694
[11] Choi, K.P. (1992). A sharp inequality for martingale transforms and the unconditional basis constant of a monotone basis in $L^p(0,1)$. *Trans. Amer. Math. Soc.* **330** 509–529. MR1034661
[12] Da Prato, G. and Zabczyk, J. (2014). *Stochastic Equations in Infinite Dimensions*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **152**. Cambridge: Cambridge Univ. Press. MR3236753
[13] Dettweiler, E. (1983). Banach space valued processes with independent increments and stochastic integration. In *Probability in Banach Spaces*, *IV* (*Oberwolfach*, 1982). *Lecture Notes in Math.* **990** 54–83. Berlin: Springer. MR0707509
[14] Dirksen, S. and Yaroslavtsev, I.S. (2017). $L^q$-valued Burkholder–Rosenthal inequalities and sharp estimates for stochastic integrals. Available at arXiv:1707.00109.
[15] Evans, L.C. and Gariepy, R.F. (1992). *Measure Theory and Fine Properties of Functions*. *Studies in Advanced Mathematics*. Boca Raton, FL: CRC Press. MR1158660
[16] Folland, G.B. (1995). *A Course in Abstract Harmonic Analysis*. *Studies in Advanced Mathematics*. Boca Raton, FL: CRC Press. MR1397028
[17] Garling, D.J.H. (1986). Brownian motion and UMD-spaces. In *Probability and Banach Spaces* (*Zaragoza*, 1985). *Lecture Notes in Math.* **1221** 36–49. Berlin: Springer. MR0875006
[18] Geiss, S., Montgomery-Smith, S. and Saksman, E. (2010). On singular integral and martingale transforms. *Trans. Amer. Math. Soc.* **362** 553–575. MR2551497
[19] Hytönen, T., van Neerven, J., Veraar, M. and Weis, L. (2016). *Analysis in Banach Spaces*. *Vol. I*. *Martingales and Littlewood–Paley Theory*. *Ergebnisse der Mathematik und Ihrer Grenzgebiete*. 3. *Folge*. *A Series of Modern Surveys in Mathematics* [*Results in Mathematics and Related Areas*. 3rd *Series*. *A Series of Modern Surveys in Mathematics*] **63**. Cham: Springer. MR3617205
[20] Itô, K. (1942). On stochastic processes. I. (Infinitely divisible laws of probability). *Jpn. J. Math.* **18** 261–301. MR0014629
[21] Jacod, J. (1979). *Calcul Stochastique et Problèmes de Martingales*. *Lecture Notes in Math.* **714**. Berlin: Springer. MR0542115
[22] Jacod, J. and Shiryaev, A.N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Berlin: Springer. MR1943877
[23] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). New York: Springer. MR1876169
[24] Karatzas, I. and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. New York: Springer. MR1121940
[25] Kingman, J.F.C. (1993). *Poisson Processes*. *Oxford Studies in Probability* **3**. New York: The Clarendon Press, Oxford Univ. Press. MR1207584
[26] McConnell, T.R. (1984). On Fourier multiplier transformations of Banach-valued functions. *Trans. Amer. Math. Soc.* **285** 739–757. MR0752501
[27] Métivier, M. (1982). *Semimartingales*: *A Course on Stochastic Processes*. *de Gruyter Studies in Mathematics* **2**. Berlin: Walter de Gruyter & Co. MR0688144

[28] Métivier, M. and Pellaumail, J. (1980). *Stochastic Integration*: *Probability and Mathematical Statistics*. New York: Academic Press. MR0578177

[29] Meyer, P.A. (1976). Un cours sur les intégrales stochastiques. In *Séminaire de Probabilités*, *X*. *Lecture Notes in Math*. **511** 245–400. Berlin: Springer. MR0501332

[30] Pisier, G. (2016). *Martingales in Banach Spaces*. *Cambridge Studies in Advanced Mathematics* **155**. Cambridge: Cambridge Univ. Press. MR3617459

[31] Protter, P.E. (2005). *Stochastic Integration and Differential Equations*. *Stochastic Modelling and Applied Probability* **21**. Berlin: Springer. Second edition. Version 2.1, corrected third printing. MR2273672

[32] Riedle, M. and van Gaans, O. (2009). Stochastic integration for Lévy processes with values in Banach spaces. *Stochastic Process*. *Appl*. **119** 1952–1974. MR2519352

[33] Rubio de Francia, J.L. (1986). Martingale and integral transforms of Banach space valued functions. In *Probability and Banach Spaces* (*Zaragoza*, 1985). *Lecture Notes in Math*. **1221** 195–222. Berlin: Springer. MR0875011

[34] Stroock, D.W. and Varadhan, S.R.S. (2006). *Multidimensional Diffusion Processes*. *Classics in Mathematics*. Berlin: Springer. Reprint of the 1997 edition. MR2190038

[35] van Neerven, J.M.A.M., Veraar, M.C. and Weis, L. (2007). Stochastic integration in UMD Banach spaces. *Ann*. *Probab*. **35** 1438–1478. MR2330977

[36] van Neerven, J.M.A.M. and Weis, L. (2005). Stochastic integration of functions with values in a Banach space. *Studia Math*. **166** 131–170. MR2109586

[37] Veraar, M. and Yaroslavtsev, I. (2016). Cylindrical continuous martingales and stochastic integration in infinite dimensions. *Electron*. *J*. *Probab*. **21** Paper No. 59. MR3563887

[38] Wang, G. (1995). Differential subordination and strong differential subordination for continuous-time martingales and related sharp inequalities. *Ann*. *Probab*. **23** 522–551. MR1334160

[39] Yaroslavtsev, I.S. (2016). Brownian representations of cylindrical continuous local martingales. *Infin*. *Dimens*. *Anal*. *Quantum Probab*. *Relat*. *Top*. To appear. Available at arXiv:1605.06946.

[40] Yaroslavtsev, I.S. (2017). Even Fourier multipliers and martingale transforms in infinite dimensions. Available at arXiv:1710.04958.

[41] Yaroslavtsev, I.S. (2017). Fourier multipliers and weak differential subordination of martingales in UMD Banach spaces. *Studia Math*. Online first version. Available at arXiv:1703.07817.

[42] Yaroslavtsev, I.S. (2017). On the martingale decompositions of Gundy, Meyer, and Yoeurp in infinite dimensions. Available at arXiv:1712.00401.

[43] Yaroslavtsev, I.S. (2019). Supplement to "Martingale decompositions and weak differential subordination in UMD Banach spaces". DOI:10.3150/18-BEJ1031SUPP.

[44] Yoeurp, Ch. (1976). Décompositions des martingales locales et formules exponentielles. In *Séminaire de Probabilités*, *X*. *Lecture Notes in Math*. **511** 432–480. Berlin: Springer. MR0451395

*Bernoulli* **25**(3), 2019, 1690–1723
https://doi.org/10.3150/18-BEJ1032

# Maximum likelihood estimators based on the block maxima method

CLÉMENT DOMBRY[1] and ANA FERREIRA[2]

[1]*Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Univ. Bourgogne Franche-Comté, 16 route de Gray, 25030 Besançon cedex, France. E-mail: clement.dombry@univ-fcomte.fr*
[2]*Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal. E-mail: anafh@tecnico.ulisboa.pt*

The extreme value index is a fundamental parameter in univariate Extreme Value Theory (EVT). It captures the tail behavior of a distribution and is central in the extrapolation beyond observations. Among other semi-parametric methods (such as the popular Hill estimator), the Block Maxima (BM) and Peaks-Over-Threshold (POT) methods are widely used for assessing the extreme value index and related normalizing constants. We provide asymptotic theory for the maximum likelihood estimators (MLE) based on the BM method for independent and identically distributed observations in the max-domain of attraction of some extreme value distribution. Our main result is the asymptotic normality of the MLE with a non-trivial bias depending on the extreme value index and on the so-called second-order parameter. Our approach combines asymptotic expansions of the likelihood process and of the empirical quantile process of block maxima. The results permit to complete the comparison of common semi-parametric estimators in EVT (MLE and probability weighted moment estimators based on the POT or BM methods) through their asymptotic variances, biases and optimal mean square errors.

*Keywords:* asymptotic normality; block maxima method; extreme value index; maximum likelihood estimator; peaks-over-threshold method; probability weighted moment estimator

## References

[1] Balkema, A.A. and de Haan, L. (1974). Residual life time at great age. *Ann. Probab.* **2** 792–804. MR0359049

[2] Beirlant, J., Goegebeur, Y., Teugels, J. and Segers, J. (2004). *Statistics of Extremes: Theory and Applications. Wiley Series in Probability and Statistics*. Chichester: Wiley. With contributions from Daniel De Waal and Chris Ferro. MR2108013

[3] Bücher, A. and Segers, J. (2017). On the maximum likelihood estimator for the generalized extreme-value distribution. *Extremes* **20** 839–872. MR3737387

[4] Bücher, A. and Segers, J. (2018). Maximum likelihood estimation for the Fréchet distribution based on block maxima extracted from a time series. *Bernoulli* **24** 1427–1462. MR3706798

[5] Coles, S. (2001). *An Introduction to Statistical Modeling of Extreme Values. Springer Series in Statistics*. London: Springer. MR1932132

[6] de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory: An Introduction. Springer Series in Operations Research and Financial Engineering*. New York: Springer. MR2234156

[7] Dietrich, D., de Haan, L. and Hüsler, J. (2002). Testing extreme value conditions. *Extremes* **5** 71–85. MR1947789

1350-7265 

[8] Dombry, C. (2015). Existence and consistency of the maximum likelihood estimators for the extreme value index within the block maxima framework. *Bernoulli* **21** 420–436. MR3322325
[9] Dombry, C. and Ferreira, A. (2019). Supplement to "Maximum likelihood estimators based on the block maxima method." DOI:10.3150/18-BEJ1032SUPP.
[10] Drees, H., de Haan, L. and Li, D. (2003). On large deviation for extremes. *Statist*. *Probab. Lett*. **64** 51–62. MR1995809
[11] Drees, H., de Haan, L. and Li, D. (2006). Approximations to the tail empirical distribution function with application to testing extreme value conditions. *J. Statist*. *Plann*. *Inference* **136** 3498–3538. MR2284668
[12] Drees, H., Ferreira, A. and de Haan, L. (2004). On maximum likelihood estimation of the extreme value index. *Ann*. *Appl*. *Probab*. **14** 1179–1201. MR2071420
[13] Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling Extremal Events*: *For Insurance and Finance*. *Applications of Mathematics* (*New York*) **33**. Berlin: Springer. MR1458613
[14] Ferreira, A. and de Haan, L. (2015). On the block maxima method in extreme value theory: PWM estimators. *Ann*. *Statist*. **43** 276–298. MR3285607
[15] Gumbel, E.J. (1958). *Statistics of Extremes*. New York: Columbia Univ. Press. MR0096342
[16] Hosking, J.R.M. and Wallis, J.R. (1987). Parameter and quantile estimation for the generalized Pareto distribution. *Technometrics* **29** 339–349. MR0906643
[17] Hosking, J.R.M., Wallis, J.R. and Wood, E.F. (1985). Estimation of the generalized extreme-value distribution by the method of probability-weighted moments. *Technometrics* **27** 251–261. MR0797563
[18] Markovich, N. (2007). *Nonparametric Analysis of Univariate Heavy-Tailed Data*: *Research and Practice*. *Wiley Series in Probability and Statistics*. Chichester: Wiley. MR2364666
[19] Novak, S.Y. (2011). *Extreme Value Methods with Applications to Finance*. *Monographs on Statistics and Applied Probability* **122**. Boca Raton, FL: CRC Press. MR2933280
[20] Pickands, J. III (1975). Statistical inference using extreme order statistics. *Ann*. *Statist*. **3** 119–131. MR0423667
[21] Prescott, P. and Walden, A.T. (1980). Maximum likelihood estimation of the parameters of the generalized extreme-value distribution. *Biometrika* **67** 723–724. MR0601119
[22] Shorack, G.R. and Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. New York: Wiley. MR0838963
[23] Smith, R.L. (1985). Maximum likelihood estimation in a class of nonregular cases. *Biometrika* **72** 67–90. MR0790201
[24] van der Vaart, A.W. (1998). *Asymptotic Statistics*. *Cambridge Series in Statistical and Probabilistic Mathematics* **3**. Cambridge: Cambridge Univ. Press. MR1652247

*Bernoulli* **25**(3), 2019, 1724–1754
https://doi.org/10.3150/18-BEJ1033

# Mixing properties and central limit theorem for associated point processes

ARNAUD POINAS[1,*], BERNARD DELYON[1,**] and FRÉDÉRIC LAVANCIER[2,3]

[1] *IRMAR, Campus de Beaulieu, Bat. 22/23, 263 avenue du Général Leclerc, 35042 Rennes, France. E-mail:* [*] *arnaud.poinas@univ-rennes1.fr;* [**] *bernard.delyon@univ-rennes1.fr*

[2] *Laboratoire de Mathématiques Jean Leray, BP 92208, 2 Rue de la Houssinière, F-44322 Nantes Cedex 03, France*

[3] *Inria, Centre Rennes Bretagne Atlantique, France. E-mail:* *Frederic.Lavancier@univ-nantes.fr*

Positively (resp. negatively) associated point processes are a class of point processes that induce attraction (resp. inhibition) between the points. As an important example, determinantal point processes (DPPs) are negatively associated. We prove $\alpha$-mixing properties for associated spatial point processes by controlling their $\alpha$-coefficients in terms of the first two intensity functions. A central limit theorem for functionals of associated point processes is deduced, using both the association and the $\alpha$-mixing properties. We discuss in detail the case of DPPs, for which we obtain the limiting distribution of sums, over subsets of close enough points of the process, of any bounded function of the DPP. As an application, we get the asymptotic properties of the parametric two-step estimator of some inhomogeneous DPPs.

*Keywords:* determinantal point process; negative association; parametric estimation; positive association; strong mixing

## References

[1] Alam, K. and Saxena, K.M.L. (1981). Positive dependence in multivariate distributions. *Comm. Statist. Theory Methods* **10** 1183–1196. MR0623526

[2] Bardenet, R. and Hardy, A. (2016). Monte Carlo with determinantal point processes. ArXiv preprint. Available at arXiv:1605.00361.

[3] Biscio, C.A.N. and Lavancier, F. (2016). Brillinger mixing of determinantal point processes and statistical applications. *Electron. J. Stat.* **10** 582–607. MR3471989

[4] Biscio, C.A.N. and Lavancier, F. (2016). Quantifying repulsiveness of determinantal point processes. *Bernoulli* **22** 2001–2028. MR3498021

[5] Biscio, C.A.N., Poinas, A. and Waagepetersen, R. (2018). A note on gaps in proofs of central limit theorems. *Statist. Probab. Lett.* **135** 7–10. MR3758254

[6] Błaszczyszyn, B. and Yogeshwaran, D. (2015). Clustering comparison of point processes, with applications to random geometric models. In *Stochastic Geometry, Spatial Statistics and Random Fields. Lecture Notes in Math.* **2120** 31–71. Cham: Springer. MR3330572

[7] Bolthausen, E. (1982). On the central limit theorem for stationary mixing random fields. *Ann. Probab.* **10** 1047–1050. MR0672305

[8] Bradley, R.C. (2005). Basic properties of strong mixing conditions. A survey and some open questions. *Probab. Surv.* **2** 107–144. Update of, and a supplement to, the 1986 original. MR2178042

[9] Bulinski, A. and Shashkin, A. (2007). *Limit Theorems for Associated Random Fields and Related Systems. Advanced Series on Statistical Science & Applied Probability* **10**. Hackensack, NJ: World Scientific Co. Pte. Ltd. MR2375106

[10] Bulinski, A.V. and Shabanovich, È. (1998). Asymptotic behavior of some functionals of positively and negatively dependent random fields. *Fundam. Prikl. Mat.* **4** 479–492. MR1801168
[11] Burton, R. and Waymire, E. (1985). Scaling limits for associated random measures. *Ann. Probab.* **13** 1267–1278. MR0806223
[12] Daley, D.J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes. Vol. I*, 2nd ed. *Probability and Its Applications* (*New York*). New York: Springer. Elementary theory and methods. MR1950431
[13] Davydov, J.A. (1968). The convergence of distributions which are generated by stationary random processes. *Teor. Veroyatn. Primen.* **13** 730–737. MR0243586
[14] Dellacherie, C. and Meyer, P.-A. (1978). *Probabilities and Potential. North-Holland Mathematics Studies* **29**. Amsterdam: North-Holland. MR0521810
[15] Deng, N., Zhou, W. and Haenggi, M. (2015). The Ginibre point process as a model for wireless networks with repulsion. *IEEE Trans. Wirel. Commun.* **1** 479–492.
[16] Doukhan, P. (1994). *Mixing: Properties and Examples. Lecture Notes in Statistics* **85**. New York: Springer. MR1312160
[17] Doukhan, P., Fokianos, K. and Li, X. (2012). On weak dependence conditions: The case of discrete valued processes. *Statist. Probab. Lett.* **82** 1941–1948. MR2970296
[18] Doukhan, P. and Louhichi, S. (1999). A new weak dependence condition and applications to moment inequalities. *Stochastic Process. Appl.* **84** 313–342. MR1719345
[19] Esary, J.D., Proschan, F. and Walkup, D.W. (1967). Association of random variables, with applications. *Ann. Math. Stat.* **38** 1466–1474. MR0217826
[20] Evans, S.N. (1990). Association and random measures. *Probab. Theory Related Fields* **86** 1–19. MR1061945
[21] Ghosh, S. (2015). Determinantal processes and completeness of random exponentials: The critical case. *Probab. Theory Related Fields* **163** 643–665. MR3418752
[22] Gomez, J.S., Vasseur, A., Vergne, A., Martins, P., Decreusefond, L. and Chen, W. (2015). A case study on regularity in cellular network deployment. *IEEE Wirel. Commun. Lett.* **4** 421–424.
[23] Guyon, X. (1995). *Random Fields on a Network. Probability and Its Applications* (*New York*). New York: Springer. Modeling, statistics, and applications, Translated from the 1992 French original by Carenne Ludeña. MR1344683
[24] Heinrich, L. (2016). On the strong Brillinger-mixing property of $\alpha$-determinantal point processes and some applications. *Appl. Math.* **61** 443–461. MR3532253
[25] Heinrich, L. and Klein, S. (2014). Central limit theorems for empirical product densities of stationary point processes. *Stat. Inference Stoch. Process.* **17** 121–138. MR3219525
[26] Hough, J.B., Krishnapur, M., Peres, Y. and Virág, B. (2009). *Zeros of Gaussian Analytic Functions and Determinantal Point Processes. University Lecture Series* **51**. Providence, RI: Amer. Math. Soc. MR2552864
[27] Ibragimov, I.A. and Linnik, Y.V. (1971). *Independent and Stationary Sequences of Random Variables*. Groningen: Wolters-Noordhoff Publishing. With a supplementary chapter by I.A. Ibragimov and V.V. Petrov, Translation from the Russian edited by J. F. C. Kingman. MR0322926
[28] Jolivet, E. (1981). Central limit theorem and convergence of empirical processes for stationary point processes. In *Point Processes and Queuing Problems* (*Colloq., Keszthely*, 1978). *Colloquia Mathematica Societatis János Bolyai* **24** 117–161. Amsterdam: North-Holland. MR0617406
[29] Kulesza, A. and Taskar, B. (2012). Determinantal point process models for machine learning. *Found. Trends Mach. Learn.* **5** 123–286.
[30] Lavancier, F., Møller, J. and Rubak, E. (2015). Determinantal point process models and statistical inference. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **77** 853–877. MR3382600

[31] Lyons, R. (2014). Determinantal probability: Basic properties and conjectures. In *Proceedings of the International Congress of Mathematicians—Seoul* 2014. *Vol*. *IV* 137–161. Seoul: Kyung Moon Sa. MR3727606
[32] Macchi, O. (1975). The coincidence approach to stochastic point processes. *Adv*. *in Appl*. *Probab*. **7** 83–122. MR0380979
[33] Miyoshi, N. and Shirai, T. (2014). A cellular network model with Ginibre configured base stations. *Adv*. *in Appl*. *Probab*. **46** 832–845. MR3254344
[34] Møller, J. and Waagepetersen, R.P. (2004). *Statistical Inference and Simulation for Spatial Point Processes*. *Monographs on Statistics and Applied Probability* **100**. Boca Raton, FL: Chapman & Hall/CRC. MR2004226
[35] Rio, E. (1993). Covariance inequalities for strongly mixing processes. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **29** 587–597. MR1251142
[36] Rosenblatt, M. (1956). A central limit theorem and a strong mixing condition. *Proc*. *Natl*. *Acad*. *Sci*. *USA* **42** 43–47. MR0074711
[37] Shirai, T. and Takahashi, Y. (2003). Random point fields associated with certain Fredholm determinants. I. Fermion, Poisson and boson point processes. *J*. *Funct*. *Anal*. **205** 414–463. MR2018415
[38] Soshnikov, A. (2000). Determinantal random point fields. *Uspekhi Mat*. *Nauk* **55** 107–160. MR1799012
[39] Waagepetersen, R. and Guan, Y. (2009). Two-step estimation for inhomogeneous spatial point processes. *J*. *R*. *Stat*. *Soc*. *Ser*. *B*. *Stat*. *Methodol*. **71** 685–702. MR2749914
[40] Yuan, M., Su, C. and Hu, T. (2003). A central limit theorem for random fields of negatively associated processes. *J*. *Theoret*. *Probab*. **16** 309–323. MR1982010

*Bernoulli* **25**(3), 2019, 1755–1769
https://doi.org/10.3150/18-BEJ1034

# Perpetual integrals via random time changes

FRANZISKA KÜHN

*Institut de Mathématiques de Toulouse, Université Paul Sabatier III Toulouse, 118 Route de Narbonne, 31062 Toulouse, France. E-mail: franziska.kuhn@math.univ-toulouse.fr*

Let $(X_t)_{t\geq 0}$ be a $d$-dimensional Feller process with symbol $q$, and let $f:\mathbb{R}^d\to(0,\infty)$ be a continuous function. In this paper, we establish a growth condition in terms of $q$ and $f$ such that the perpetual integral

$$\int_0^\infty f(X_s)\,ds$$

is infinite almost surely. The result applies, in particular, if $(X_t)_{t\geq 0}$ is a Lévy process. The key idea is to approach perpetuals integrals via random time changes. As a by-product of the proof, a sufficient condition for the non-explosion of solutions to martingale problems is obtained. Moreover, we establish a condition which ensures that the random time change of a Feller process is a conservative $C_b$-Feller process.

*Keywords:* conservativeness; Feller process; Lévy process; perpetual integral; random time change

## References

[1] Böttcher, B., Schilling, R. and Wang, J. (2013). *Lévy-Type Processes*: *Construction*, *Approximation and Sample Path Properties*. *Lecture Notes in Math*. **2099**. Cham: Springer. MR3156646

[2] Döring, L. and Kyprianou, A.E. (2016). Perpetual integrals for Lévy processes. *J. Theoret. Probab.* **29** 1192–1198. MR3540494

[3] Engelbert, H.J. and Senf, T. (1991). On functionals of a Wiener process with drift and exponential local martingales. In *Stochastic Processes and Related Topics* (*Georgenthal*, 1990) (M. Dozzi, H.J. Engelbert and D. Nualart, eds.). *Math*. *Res*. **61** 45–58. Berlin: Akademie-Verlag. MR1127879

[4] Erickson, K.B. and Maller, R.A. (2005). Generalised Ornstein–Uhlenbeck processes and the convergence of Lévy integrals. In *Séminaire de Probabilités XXXVIII* (M. Émery, M. Ledoux and M. Yor, eds.). *Lecture Notes in Math*. **1857** 70–94. Springer, Berlin. MR2126967

[5] Ethier, S.N. and Kurtz, T.G. (1986). *Markov Processes*: *Characterization and Convergence*. New York: Wiley. MR0838085

[6] Helland, I.S. (1978). Continuity of a class of random time transformations. *Stochastic Process*. *Appl*. **7** 79–99. MR0488203

[7] Itô, K. (2004). *Stochastic Processes*. Berlin: Springer.

[8] Khoshnevisan, D., Salminen, P. and Yor, M. (2006). A note on a.s. finiteness of perpetual integral functionals of diffusions. *Electron*. *Commun*. *Probab*. **11** 108–117. MR2231738

[9] Kolokoltsov, V.N. (2011). *Markov Processes*, *Semigroups and Generators*. *De Gruyter Studies in Mathematics* **38**. Berlin: de Gruyter. MR2780345

[10] Kühn, F. Solutions of Lévy-driven SDEs with unbounded coefficients as Feller processes. *Proc*. *Amer*. *Math*. *Soc*. To appear. Available at arXiv:1610.02286.

[11] Kühn, F. (2016). Probability and Heat Kernel Estimates for Lévy(-Type) Processes. Ph.D. thesis, Technische Universität Dresden. Available at http://nbn-resolving.de/urn:nbn:de:bsz:14-qucosa-214839.

1350-7265 

[12] Kühn, F. (2017). *Lévy-Type Processes*: *Moments*, *Construction and Heat Kernel Estimates*. *Lecture Notes in Math*. **2187**. Cham: Springer. MR3701414

[13] Kühn, F. (2018). On martingale problems and Feller processes. *Electron*. *J*. *Probab*. **23** 1–18.

[14] Lamperti, J. (1967). On random time substitutions and the Feller property. In *Markov Processes and Potential Theory* (*Proc Sympos*. *Math*. *Res*. *Center*, *Madison*, *Wis*., 1967) 87–101. Wiley, New York. MR0230370

[15] Salminen, P. and Yor, M. (2005). Perpetual integral functionals as hitting and occupation times. *Electron*. *J*. *Probab*. **10** 371–419. MR2147313

[16] Salminen, P. and Yor, M. (2005). Properties of perpetual integral functionals of Brownian motion with drift. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **41** 335–347. MR2139023

[17] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. MR1739520

[18] Schilling, R.L. (1998). Conservativeness and extensions of Feller semigroups. *Positivity* **2** 239–256. MR1653474

[19] Schilling, R.L. and Vondraček, Z. (2017). Absolute continuity and singularity of probability measures induced by a purely discontinuous Girsanov transform of a stable process. *Trans*. *Amer*. *Math*. *Soc*. **369** 1547–1577. MR3581212

[20] Volkonskiĭ, V.A. (1958). Random substitution of time in strong Markov processes. *Teor*. *Veroyatn*. *Primen*. **3** 332–350. MR0100919

[21] Wang, J. (2011). Stability of Markov processes generated by Lévy-type operators. *Chinese Ann*. *Math*. *Ser*. *A* **32** 33–50. MR2663819

*Bernoulli* **25**(3), 2019, 1770–1793
https://doi.org/10.3150/18-BEJ1035

# Uniform behaviors of random polytopes under the Hausdorff metric

VICTOR-EMMANUEL BRUNEL

*Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA. E-mail: veb@mit.edu*

We study the Hausdorff distance between a random polytope, defined as the convex hull of i.i.d. random points, and the convex hull of the support of their distribution. As particular examples, we consider uniform distributions on convex bodies, densities that decay at a certain rate when approaching the boundary of a convex body, projections of uniform distributions on higher dimensional convex bodies and uniform distributions on the boundary of convex bodies. We essentially distinguish two types of convex bodies: those with a smooth boundary and polytopes. In the case of uniform distributions, we prove that, in some sense, the random polytope achieves its best statistical accuracy under the Hausdorff metric when the support has a smooth boundary and its worst statistical accuracy when the support is a polytope. This is somewhat surprising, since the exact opposite is true under the Nikodym metric. We prove rate optimality of most our results in a minimax sense. In the case of uniform distributions, we extend our results to a rescaled version of the Hausdorff metric. We also tackle the estimation of functionals of the support of a distribution such as its mean width and its diameter. Finally, we show that high dimensional random polytopes can be approximated with simple polyhedral representations that significantly decrease their computational complexity without affecting their statistical accuracy.

*Keywords:* computational geometry; convex bodies; convex hull; deviation inequality; Hausdorff metric; high dimension; minimax estimation; random polytope

## References

[1] Affentranger, F. and Wieacker, J.A. (1991). On the convex hull of uniform random points in a simple $d$-polytope. *Discrete Comput. Geom.* **6** 291–305. MR1098810
[2] Baldin, N. and Reiß, M. (2016). Unbiased estimation of the volume of a convex body. *Stochastic Process. Appl.* **126** 3716–3732. MR3565474
[3] Bárány, I. (1989). Intrinsic volumes and $f$-vectors of random polytopes. *Math. Ann.* **285** 671–699. MR1027765
[4] Bárány, I. (2005). Corrigendum: "Random polytopes in smooth convex bodies" [*Mathematika* **39** (1992) 81–92 MR1176473]. *Mathematika* **51** 31.
[5] Bárány, I. and Buchta, C. (1993). Random polytopes in a convex polytope, independence of shape, and concentration of vertices. *Math. Ann.* **297** 467–497. MR1245400
[6] Bárány, I., Fodor, F. and Vígh, V. (2010). Intrinsic volumes of inscribed random polytopes in smooth convex bodies. *Adv. in Appl. Probab.* **42** 605–619. MR2779551
[7] Bárány, I. and Larman, D.G. (1988). Convex bodies, economic cap coverings, random polytopes. *Mathematika* **35** 274–291. MR0986636
[8] Böröczky, K.J., Fodor, F., Reitzner, M. and Vígh, V. (2009). Mean width of random polytopes in a reasonably smooth convex body. *J. Multivariate Anal.* **100** 2287–2295. MR2560369

1350-7265 

[9] Böröczky, K.J. Jr., Hoffmann, L.M. and Hug, D. (2008). Expectation of intrinsic volumes of random polytopes. *Period*. *Math*. *Hungar*. **57** 143–164. MR2469601

[10] Brunel, V.-E. Uniform deviation and moment inequalities for random polytopes with general densities in arbitrary convex bodies. Preprint. Available at arXiv:1704.01620.

[11] Brunel, V.-E. (2016). Adaptive estimation of convex and polytopal density support. *Probab*. *Theory Related Fields* **164** 1–16. MR3449384

[12] Brunel, V.-E. (2018). Concentration of the empirical level sets of Tukey's halfspace depth. Probability and Related Fields. To appear. Available at arXiv:1605.09456.

[13] Buchta, C. and Müller, J. (1984). Random polytopes in a ball. *J*. *Appl*. *Probab*. **21** 753–762. MR0766813

[14] Chazelle, B. (1993). An optimal convex hull algorithm in any fixed dimension. *Discrete Comput*. *Geom*. **10** 377–409. MR1243335

[15] Cuevas, A. (2009). Set estimation: Another bridge between statistics and geometry. *Bol*. *Estad*. *Investig*. *Oper*. **25** 71–85. MR2750781

[16] Dümbgen, L. and Walther, G. (1996). Rates of convergence for random approximations of convex sets. *Adv*. *in Appl*. *Probab*. **28** 384–393. MR1387882

[17] Dwyer, R.A. (1988). On the convex hull of random points in a polytope. *J*. *Appl*. *Probab*. **25** 688–699. MR0967853

[18] Efron, B. (1965). The convex hull of a random set of points. *Biometrika* **52** 331–343. MR0207004

[19] Fresen, D.J. and Vitale, R.A. (2014). Concentration of random polytopes around the expected convex hull. *Electron*. *Commun*. *Probab*. **19** 8. MR3254738

[20] Gayraud, G. (1997). Estimation of functionals of density support. *Math*. *Methods Statist*. **6** 26–46. MR1456645

[21] Gerchinovitz, S., Ménard, P. and Stoltz, G. (2017). Fano's inequality for random variables. Preprint. Available at arXiv:1702.05985.

[22] Glasauer, S. and Schneider, R. (1996). Asymptotic approximation of smooth convex bodies by polytopes. *Forum Math*. **8** 363–377. MR1387701

[23] Groemer, H. (1974). On the mean value of the volume of a random polytope in a convex set. *Arch*. *Math*. (*Basel*) **25** 86–90. MR0341286

[24] Guntuboyina, A. (2012). Optimal rates of convergence for convex set estimation from support functions. *Ann*. *Statist*. **40** 385–411. MR3014311

[25] Hug, D. (2013). Random polytopes. In *Stochastic Geometry*, *Spatial Statistics and Random Fields*. *Lecture Notes in Math*. **2068** 205–238. Springer, Heidelberg. MR3059649

[26] Kolmogorov, A.N. and Tihomirov, V.M. (1959). $\varepsilon$-entropy and $\varepsilon$-capacity of sets in function spaces. *Uspekhi Mat*. *Nauk* **14** 3–86. MR0112032

[27] Korostelëv, A.P., Simar, L. and Tsybakov, A.B. (1995). Efficient estimation of monotone boundaries. *Ann*. *Statist*. **23** 476–489. MR1332577

[28] Korostelev, A.P., Simar, L. and Tsybakov, A.B. (1995). On estimation of monotone and convex boundaries. *Publ*. *Inst*. *Stat*. *Univ*. *Paris* **39** 3–18. MR1744393

[29] Korostelëv, A.P. and Tsybakov, A.B. (1993). *Minimax Theory of Image Reconstruction*. *Lecture Notes in Statistics* **82**. New York: Springer. MR1226450

[30] Korostelëv, A.P. and Tsybakov, A.B. (1994). Asymptotic efficiency of the estimation of a convex set. *Problemy Peredachi Informatsii* **30** 33–44. MR1310060

[31] Leichtweiss, K. (1959). Über die affine Exzentrizität konvexer Körper. *Arch*. *Math*. **10** 187–199. MR0107843

[32] Molchanov, I. (2013). Foundations of stochastic geometry and theory of random sets. In *Stochastic Geometry*, *Spatial Statistics and Random Fields*. *Lecture Notes in Math*. **2068** 1–20. Springer, Heidelberg. MR3059643

[33] Moore, M. (1984). On the estimation of a convex set. *Ann. Statist.* **12** 1090–1099. MR0751296
[34] Müller, J.S. (1989). On the mean width of random polytopes. *Probab. Theory Related Fields* **82** 33–37. MR0997430
[35] Pateiro-Lopez, B. (2008). Set estimation under convexity type restrictions. Ph.D. Thesis, University of Santiago de Compostela.
[36] Reisner, S., Schütt, C. and Werner, E. (2001). Dropping a vertex or a facet from a convex polytope. *Forum Math.* **13** 359–378. MR1831090
[37] Reitzner, M. (2003). Random polytopes and the Efron–Stein jackknife inequality. *Ann. Probab.* **31** 2136–2166. MR2016615
[38] Reitzner, M. (2004). Stochastic approximation of smooth convex bodies. *Mathematika* **51** 11–29. MR2220208
[39] Reitzner, M. (2010). Random polytopes. In *New Perspectives in Stochastic Geometry* 45–76. Oxford Univ. Press, Oxford. MR2654675
[40] Rényi, A. and Sulanke, R. (1963). Über die konvexe Hülle von $n$ zufällig gewählten Punkten. *Z. Wahrsch. Verw. Gebiete* **2** 75–84. MR0156262
[41] Rényi, A. and Sulanke, R. (1964). Über die konvexe Hülle von $n$ zufällig gewählten Punkten. II. *Z. Wahrsch. Verw. Gebiete* **3** 138–147. MR0169139
[42] Schneider, R. (1987). Approximation of convex bodies by random polytopes. *Aequationes Math.* **32** 304–310. MR0900710
[43] Schneider, R. (2008). Recent results on random polytopes. *Boll. Unione Mat. Ital.* (9) **1** 17–39. MR2387995
[44] Schneider, R. (2014). *Convex Bodies: The Brunn–Minkowski Theory*, expanded ed. *Encyclopedia of Mathematics and Its Applications* **151**. Cambridge: Cambridge Univ. Press. MR3155183
[45] Schneider, R. and Wieacker, J.A. (1980). Random polytopes in a convex body. *Z. Wahrsch. Verw. Gebiete* **52** 69–73. MR0568260
[46] Schütt, C. (1994). Random polytopes and affine surface area. *Math. Nachr.* **170** 227–249. MR1302377
[47] Thäle, C. (2008). 50 years sets with positive reach – a survey. *Surv. Math. Appl.* **3** 123–165. MR2443192
[48] Tsybakov, A.B. (2009). *Introduction to Nonparametric Estimation*. *Springer Series in Statistics*. New York: Springer. Revised and extended from the 2004 French original, Translated by Vladimir Zaiats. MR2724359
[49] Vu, V.H. (2005). Sharp concentration of random polytopes. *Geom. Funct. Anal.* **15** 1284–1318. MR2221249

*Bernoulli* **25**(3), 2019, 1794–1815
https://doi.org/10.3150/18-BEJ1036

# On the isoperimetric constant, covariance inequalities and $L_p$-Poincaré inequalities in dimension one

ADRIEN SAUMARD[1] and JON A. WELLNER[2]

[1]*CREST, Ensai, Université Bretagne Loire, Rennes, France. E-mail: asaumard@gmail.com*
[2]*Department of Statistics, University of Washington, Seattle, WA 98195-4322, USA. E-mail: jaw@stat.washington.edu*

First, we derive in dimension one a new covariance inequality of $L_1 - L_\infty$ type that characterizes the isoperimetric constant as the best constant achieving the inequality. Second, we generalize our result to $L_p - L_q$ bounds for the covariance. Consequently, we recover Cheeger's inequality without using the co-area formula. We also prove a generalized weighted Hardy type inequality that is needed to derive our covariance inequalities and that is of independent interest. Finally, we explore some consequences of our covariance inequalities for $L_p$-Poincaré inequalities and moment bounds. In particular, we obtain optimal constants in general $L_p$-Poincaré inequalities for measures with finite isoperimetric constant, thus generalizing in dimension one Cheeger's inequality, which is a $L_p$-Poincaré inequality for $p = 2$, to any real $p \geq 1$.

*Keywords:* Cheeger's inequality; covariance formula; covariance inequality; isoperimetric constant; moment bounds; Poincaré inequality

## References

[1] Adamczak, R., Latała, R., Litvak, A.E., Oleszkiewicz, K., Pajor, A. and Tomczak-Jaegermann, N. (2014). A short proof of Paouris' inequality. *Canad. Math. Bull.* **57** 3–8. MR3150710

[2] Adamczak, R., Latała, R., Litvak, A.E., Pajor, A. and Tomczak-Jaegermann, N. (2014). Tail estimates for norms of sums of log-concave random vectors. *Proc. Lond. Math. Soc.* (3) **108** 600–637. MR3180591

[3] Ané, C., Blachère, S., Chafaï, D., Fougères, P., Gentil, I., Malrieu, F., Roberto, C. and Scheffer, G. (2000). *Sur les Inégalités de Sobolev Logarithmiques*. *Panoramas et Synthèses* [*Panoramas and Syntheses*] **10**. Paris: Société Mathématique de France. With a preface by Dominique Bakry and Michel Ledoux. MR1845806

[4] Block, H.W. and Fang, Z.B. (1988). A multivariate extension of Hoeffding's lemma. *Ann. Probab.* **16** 1803–1820. MR0958217

[5] Bobkov, S. and Ledoux, M. (2014). One-dimensional empirical measures, order statistics and Kantorovich transport distances. Preprint.

[6] Bobkov, S.G. and Götze, F. (1999). Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. *J. Funct. Anal.* **163** 1–28. MR1682772

[7] Bobkov, S.G., Götze, F. and Houdré, C. (2001). On Gaussian and Bernoulli covariance representations. *Bernoulli* **7** 439–451. MR1836739

[8] Bobkov, S.G. and Houdré, C. (1997). Isoperimetric constants for product probability measures. *Ann. Probab.* **25** 184–205. MR1428505
[9] Bobkov, S.G. and Houdré, C. (1997). Some connections between isoperimetric and Sobolev-type inequalities. *Mem. Amer. Math. Soc.* **129** viii + 111. MR1396954
[10] Bobkov, S.G. and Zegarlinski, B. (2005). Entropy bounds and isoperimetry. *Mem. Amer. Math. Soc.* **176** x + 69. MR2146071
[11] Borell, C. (1974). Convex measures on locally convex spaces. *Ark. Mat.* **12** 239–252. MR0388475
[12] Bubeck, S. and Eldan, R. (2014). The entropic barrier: A simple and optimal universal self-concordant barrier. Preprint. Available at arXiv:1412.1587.
[13] Carlen, E.A., Cordero-Erausquin, D. and Lieb, E.H. (2013). Asymmetric covariance estimates of Brascamp–Lieb type and related inequalities for log-concave measures. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** 1–12. MR3060145
[14] Fougères, P. (2005). Spectral gap for log-concave probability measures on the real line. In *Séminaire de Probabilités XXXVIII. Lecture Notes in Math.* **1857** 95–123. Berlin: Springer. MR2126968
[15] Fradelizi, M. and Guédon, O. (2006). A generalized localization theorem and geometric inequalities for convex bodies. *Adv. Math.* **204** 509–529. MR2249622
[16] Hoeffding, W. (1994). *The Collected Works of Wassily Hoeffding. Springer Series in Statistics: Perspectives in Statistics.* New York: Springer. Edited and with a preface by N.I. Fisher and P.K. Sen. MR1307621
[17] Höffding, W. (1940). Maszstabinvariante Korrelationstheorie. *Schr. Math. Inst. U. Inst. Angew. Math. Univ. Berlin* **5** 181–233. MR0004426
[18] Houdré, C. (2002). Remarks on deviation inequalities for functions of infinitely divisible random vectors. *Ann. Probab.* **30** 1223–1237. MR1920106
[19] Houdré, C. and Marchal, P. (2004). On the concentration of measure phenomenon for stable and related random vectors. *Ann. Probab.* **32** 1496–1508. MR2060306
[20] Latała, R. (2011). Weak and strong moments of random vectors. In *Marcinkiewicz Centenary Volume. Banach Center Publ.* **95** 115–121. Warsaw: Polish Acad. Sci. Inst. Math. MR2918093
[21] Latała, R. (2017). On some problems concerning log-concave random vectors. In *Convexity and Concentration* (E. Carlen, M. Madiman and E.M. Werner, eds.) 525–539. New York, NY: Springer.
[22] Latała, R. and Strzelecka, M. (2016). Weak and strong moments of $\ell_r$-norms of log-concave vectors. *Proc. Amer. Math. Soc.* **144** 3597–3608. MR3503729
[23] Latała, R. and Wojtaszczyk, J.O. (2008). On the infimum convolution inequality. *Studia Math.* **189** 147–187. MR2449135
[24] Ledoux, M. (2004). Spectral gap, logarithmic Sobolev constant, and geometric bounds. In *Surveys in Differential Geometry. Vol. IX. Surv. Differ. Geom.* **9** 219–240. Somerville, MA: Int. Press. MR2195409
[25] Menz, G. and Otto, F. (2013). Uniform logarithmic Sobolev inequalities for conservative spin systems with super-quadratic single-site potential. *Ann. Probab.* **41** 2182–2224. MR3098070
[26] Miclo, L. (1999). An example of application of discrete Hardy's inequalities. *Markov Process. Related Fields* **5** 319–330. MR1710983
[27] Milman, E. (2009). On the role of convexity in isoperimetry, spectral gap and concentration. *Invent. Math.* **177** 1–43. MR2507637
[28] Paouris, G. (2006). Concentration of mass on convex bodies. *Geom. Funct. Anal.* **16** 1021–1049. MR2276533
[29] Saumard, A. and Wellner, J.A. (2014). Log-concavity and strong log-concavity: A review. *Stat. Surv.* **8** 45–114. MR3290441
[30] Saumard, A. and Wellner, J.A. (2017). On the isoperimetric constant, covariance inequalities and $L_p$-Poincaré inequalities in dimension one. Technical report, Department of Statistics, University of Washington. Available at arXiv:1711.00668.

[31] Saumard, A. and Wellner, J.A. (2018). Efron's monotonicity property for measures on $\mathbb{R}^2$. *J. Multivariate Anal*. **166** 212–224. MR3799644

*Bernoulli* **25**(3), 2019, 1816–1837
https://doi.org/10.3150/18-BEJ1037

# A one-sample test for normality with kernel methods

JÉRÉMIE KELLNER* and ALAIN CELISSE**

*Laboratoire de Mathématiques, UMR 8524 CNRS – Université Lille 1 – MODAL team-project Inria, 59655, Villeneuve d'Ascq Cedex, France.*
*E-mail:* *jeremie.kellner@ed.univ-lille1.fr; **alain.celisse@math.univ-lille1.fr

We propose a new one-sample test for normality in a Reproducing Kernel Hilbert Space (RKHS). Namely, we test the null-hypothesis of belonging to a given family of Gaussian distributions. Hence, our procedure may be applied either to test data for normality or to test parameters (mean and covariance) if data are assumed Gaussian. Our test is based on the same principle as the MMD (Maximum Mean Discrepancy) which is usually used for two-sample tests such as homogeneity or independence testing. Our method makes use of a special kind of parametric bootstrap (typical of goodness-of-fit tests) which is computationally more efficient than standard parametric bootstrap. Moreover, an upper bound for the Type-II error highlights the dependence on influential quantities. Experiments illustrate the practical improvement allowed by our test in high-dimensional settings where common normality tests are known to fail. We also consider an application to covariance rank selection through a sequential procedure.

*Keywords:* kernel methods; maximum mean discrepancy; normality test; parametric bootstrap; reproducing kernel hilbert space

## References

[1] Aronszajn, N. (1950). Theory of reproducing kernels. *Trans. Amer. Math. Soc.* **68** 337–404. MR0051437
[2] Bien, J. and Tibshirani, R.J. (2011). Sparse estimation of a covariance matrix. *Biometrika* **98** 807–820. MR2860325
[3] Blanchard, G., Sugiyama, M., Kawanabe, M., Spokoiny, V. and Muller, K.-R. (2006). Non-Gaussian component analysis: A semi-parametric framework for linear dimension reduction. In *NIPS*.
[4] Bouveyron, C., Fauvel, M. and Girard, S. (2015). Kernel discriminant analysis and clustering with parsimonious Gaussian process models. *Stat. Comput.* **25** 1143–1162. MR3401878
[5] Brunel, É., Mas, A. and Roche, A. (2016). Non-asymptotic adaptive prediction in functional linear models. *J. Multivariate Anal.* **143** 208–232. MR3431429
[6] Burke, M.D. (2000). Multivariate tests-of-fit and uniform confidence bands using a weighted bootstrap. *Statist. Probab. Lett.* **46** 13–20. MR1730878
[7] Cardot, H. and Johannes, J. (2010). Thresholding projection estimators in functional linear models. *J. Multivariate Anal.* **101** 395–408. MR2564349
[8] Choi, Y., Taylor, J. and Tibshirani, R. (2017). Selecting the number of principal components: Estimation of the true rank of a noisy matrix. *Ann. Statist.* **45** 2590–2617. MR3737903
[9] Christmann, A. and Steinwart, I. (2010). Universal kernels on non-standard input spaces. In *Advances in Neural Information Processing Systems* 406–414.

1350-7265 

[10] Cuesta-Albertos, J.A., del Barrio, E., Fraiman, R. and Matrán, C. (2007). The random projection method in goodness of fit for functional data. *Comput*. *Statist*. *Data Anal*. **51** 4814–4831. MR2364542
[11] Diederichs, E., Juditsky, A., Nemirovski, A. and Spokoiny, V. (2013). Sparse non Gaussian component analysis by semidefinite programming. *Mach*. *Learn*. **91** 211–238. MR3046905
[12] Diederichs, E., Juditsky, A., Spokoiny, V. and Schütte, C. (2010). Sparse non-Gaussian component analysis. *IEEE Trans*. *Inform*. *Theory* **56** 3033–3047. MR2683454
[13] Frigyik, B.A., Srivastava, S. and Gupta, M.R. (2008). An introduction to functional derivatives. Technical Report, Dept. Electr. Eng., Univ. Washington, Seattle, WA.
[14] Fukumizu, K., Gretton, A., Sun, X. and Schölkopf, B. (2007). Kernel measures of conditional dependence. In *NIPS* **20** 489–496.
[15] Fukumizu, K., Sriperumbudur, B., Gretton, A. and Schölkopf, B. (2009). Characteristic Kernels on groups and semigroups. In *NIPS*.
[16] Gretton, A., Borgwardt, K., Rasch, M., Schoelkopf, B. and Smola, A. (2007). A kernel method for the two-sample-problem. In *Advances in Neural Information Processing Systems* (B. Schoelkopf, J. Platt and T. Hoffman, eds.) **19** 513–520. Cambridge: MIT Press.
[17] Gretton, A., Borgwardt, K.M., Rasch, M.J., Schölkopf, B. and Smola, A. (2012). A kernel two-sample test. *J*. *Mach*. *Learn*. *Res*. **13** 723–773. MR2913716
[18] Gretton, A., Fukumizu, K., Harchaoui, Z. and Sriperumbudur, B.K. (2009). A fast, consistent kernel two-sample test. In *NIPS*.
[19] Gretton, A., Fukumizu, K., Teo, C.H., Song, L., Schölkopf, B. and Smola, A.J. (2007). A kernel statistical test of independence. In *NIPS* 21.
[20] Gretton, A., Sejdinovic, D., Strathmann, H., Balakrishnan, S., Pontil, M., Fukumizu, K. and Sriperumbudur, B.K. (2012). Optimal kernel choice for large-scale two-sample tests. In *Advances in Neural Information Processing Systems* 1205–1213.
[21] Henze, N. and Zirkler, B. (1990). A class of invariant consistent tests for multivariate normality. *Comm*. *Statist*. *Theory Methods* **19** 3595–3617. MR1089501
[22] Hoffmann-Jørgensen, J. and Pisier, G. (1976). The law of large numbers and the central limit theorem in Banach spaces. *Ann. Probab.* **4** 587–599. MR0423451
[23] Josse, J. and Husson, F. (2012). Selecting the number of components in principal component analysis using cross-validation approximations. *Comput*. *Statist*. *Data Anal*. **56** 1869–1879. MR2892383
[24] Kellner, J. and Celisse, A. (2019). Supplement to “A one-sample test for normality with kernel methods.” DOI:10.3150/18-BEJ1037SUPP.
[25] Kojadinovic, I. and Yan, J. (2012). Goodness-of-fit testing based on a weighted bootstrap: A fast large-sample alternative to the parametric bootstrap. *Canad*. *J*. *Statist*. **40** 480–500. MR2968394
[26] Lehmann, E.L. and Romano, J.P. (2005). *Testing Statistical Hypotheses*, 3rd ed. New York: Springer. MR2135927
[27] Mardia, K.V. (1970). Measures of multivariate skewness and kurtosis with applications. *Biometrika* **57** 519–530. MR0397994
[28] Ratsimalahelo, Z. (2003). Strongly consistent determination of the rank of matrix. *Econometrics*.
[29] Reddi, S., Ramdas, A., Poczos, B., Singh, A. and Wasserman, L. (2015). On the high dimensional power of a linear-time two sample test under mean-shift alternatives. *J*. *Mach*. *Learn*. *Res*. 772–780.
[30] Robin, J.-M. and Smith, R.J. (2000). Tests of rank. *Econometric Theory* **16** 151–175. MR1763430
[31] Roth, V. (2006). Kernel Fisher discriminants for outlier detection. *Neural Comput*. **18** 942–960. MR2204796
[32] Roweis, S. (1998). EM algorithms for PCA and SPCA. In *Advances in Neural Information Processing Systems* 626–632. MIT Press.
[33] Schölkopf, B., Smola, A. and Müller, K.-R. (1997). Nonlinear component analysis as a kernel eigenvalue problem. *Neural Comput*. **10** 1299–1319.

[34] Sejdinovic, D., Sriperumbudur, B., Gretton, A. and Fukumizu, K. (2013). Equivalence of distance-based and RKHS-based statistics in hypothesis testing. *Ann. Statist.* **41** 2263–2291. MR3127866

[35] Sriperumbudur, B.K., Gretton, A., Fukumizu, K., Schölkopf, B. and Lanckriet, G.R.G. (2010). Hilbert space embeddings and metrics on probability measures. *J. Mach. Learn. Res.* **11** 1517–1561. MR2645460

[36] Srivastava, M.S., Katayama, S. and Kano, Y. (2013). A two sample test in high dimensional data. *J. Multivariate Anal.* **114** 349–358. MR2993891

[37] Stute, W., Gonzáles Manteiga, W. and Presedo Quindimil, M. (1993). Bootstrap based goodness-of-fit tests. *Metrika* **40** 243–256. MR1235086

[38] Svantesson, T. and Wallace, J.W. (2003). Tests for assessing multivariate normality and the covariance structure of MIMO data. In *Acoustics, Speech, and Signal Processing*, 2003. *Proceedings.* (*ICASSP'*03). 2003 *IEEE International Conference on* **4** IV–656. IEEE.

[39] Székely, G.J. and Rizzo, M.L. (2005). A new test for multivariate normality. *J. Multivariate Anal.* **93** 58–80. MR2119764

[40] Zwald, L. (2005). Performances d'Algorithmes Statistiques d'Apprentissage: "Kernel Projection Machine" et Analyse en Composantes Principales à Noyaux. Ph.D. thesis, Université Paris XI U.F.R. Scientifique d'Orsay.

*Bernoulli* **25**(3), 2019, 1838–1869
https://doi.org/10.3150/18-BEJ1038

# Central limit theorem for linear spectral statistics of large dimensional separable sample covariance matrices

ZHIDONG BAI[1], HUIQIN LI[2] and GUANGMING PAN[3]

[1]*KLASMOE and School of Mathematics & Statistics, Northeast Normal University, Changchun, P.R.C., 130024. E-mail: baizd@nenu.edu.cn*
[2]*School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, P.R.C., 221116. E-mail: huiqinli@jsnu.edu.cn*
[3]*Division of Mathematical Sciences, School of Physical and Mathmatical Sciences, Nanyang Technological University, Singapore 637371. E-mail: GMPAN@ntu.edu.sg*

Suppose that $\mathbf{X}_n = (x_{jk})$ is $N \times n$ whose elements are independent complex variables with mean zero, variance 1. The separable sample covariance matrix is defined as $\mathbf{B}_n = \frac{1}{N}\mathbf{T}_{2n}^{1/2}\mathbf{X}_n\mathbf{T}_{1n}\mathbf{X}_n^*\mathbf{T}_{2n}^{1/2}$ where $\mathbf{T}_{1n}$ is a Hermitian matrix and $\mathbf{T}_{2n}^{1/2}$ is a Hermitian square root of the nonnegative definite Hermitian matrix $\mathbf{T}_{2n}$. Its linear spectral statistics (LSS) are shown to have Gaussian limits when $n/N$ approaches a positive constant under some conditions.

*Keywords:* central limit theorem; linear spectral statistics; random matrix theory; separable sample covariance matrix

## References

[1] Bai, Z.D., Li, H. and Pan, G. (2019). Supplement to "Central limit theorem for linear spectral statistics of large dimensional separable sample covariance matrices." DOI:10.3150/18-BEJ1038SUPP.
[2] Bai, Z.D. and Silverstein, J.W. (2004). CLT for linear spectral statistics of large-dimensional sample covariance matrices. *Ann. Probab.* **32** 553–605. MR2040792
[3] Bai, Z.D. and Yin, Y.Q. (1993). Limit of the smallest eigenvalue of a large-dimensional sample covariance matrix. *Ann. Probab.* **21** 1275–1294. MR1235416
[4] Boutet de Monvel, A., Khorunzhy, A. and Vasilchuk, V. (1996). Limiting eigenvalue distribution of random matrices with correlated entries. *Markov Process. Related Fields* **2** 607–636. MR1431189
[5] Couillet, R. and Hachem, W. (2014). Analysis of the limiting spectral measure of large random matrices of the separable covariance type. *Random Matrices Theory Appl.* **3** 1450016, 23. MR3279621
[6] Götze, F., Naumov, A. and Tikhomirov, A. (2017). Distribution of linear statistics of singular values of the product of random matrices. *Bernoulli* **23** 3067–3113. MR3654799
[7] Jiang, T. and Yang, F. (2013). Central limit theorems for classical likelihood ratio tests for high-dimensional normal distributions. *Ann. Statist.* **41** 2029–2074. MR3127857
[8] Jonsson, D. (1982). Some limit theorems for the eigenvalues of a sample covariance matrix. *J. Multivariate Anal.* **12** 1–38. MR0650926
[9] Ledoit, O. and Wolf, M. (2002). Some hypothesis tests for the covariance matrix when the dimension is large compared to the sample size. *Ann. Statist.* **30** 1081–1102. MR1926169

[10] Li, P., Paul, D., Narasimhan, R. and Cioffi, J. (2006). On the distribution of SINR for the MMSE MIMO receiver and performance analysis. *IEEE Trans. Inform. Theory* **52** 271–286. MR2237352

[11] Nadakuditi, R.R. and Edelman, A. (2008). Sample eigenvalue based detection of high-dimensional signals in white noise using relatively few samples. *IEEE Trans. Signal Process.* **56** 2625–2638. MR1500236

[12] Najim, J. and Yao, J. (2016). Gaussian fluctuations for linear spectral statistics of large random covariance matrices. *Ann. Appl. Probab.* **26** 1837–1887. MR3513608

[13] Onatski, A. (2010). Determining the number of factors from empirical distribution of eigenvalues. *Rev. Econ. Stat.* **92** 1004–1016.

[14] Pan, G.M. and Zhou, W. (2008). Central limit theorem for signal-to-interference ratio of reduced rank linear receiver. *Ann. Appl. Probab.* **18** 1232–1270. MR2418244

[15] Paul, D. and Silverstein, J.W. (2009). No eigenvalues outside the support of the limiting empirical spectral distribution of a separable covariance matrix. *J. Multivariate Anal.* **100** 37–57. MR2460475

[16] Shcherbina, M. (2011). Central limit theorem for linear eigenvalue statistics of the Wigner and sample covariance random matrices. *Zh. Mat. Fiz. Anal. Geom.* **7** 176–192, 197, 199. MR2829615

[17] Tulino, A.M. and Verdu, S. (2004). Random matrix theory and wireless communications. *Found. Trends Commun. Inf. Theory* **1** 1–182.

[18] Wishart, J. (1928). The generalised product moment distribution in samples from a normal multivariate population. *Biometrika* **20** 32–52.

[19] Yao, J., Zheng, S. and Bai, Z.D. (2015). *Large Sample Covariance Matrices and High-Dimensional Data Analysis. Cambridge Series in Statistical and Probabilistic Mathematics* **39**. New York: Cambridge Univ. Press. MR3468554

[20] Yin, Y.Q., Bai, Z.D. and Krishnaiah, P.R. (1988). On the limit of the largest eigenvalue of the large-dimensional sample covariance matrix. *Probab. Theory Related Fields* **78** 509–521. MR0950344

[21] Zhang, L.X. (2006). Spectral analysis of large dimensional random matrices. PhD thesis, National Univ. Singapore.

*Bernoulli* **25**(3), 2019, 1870–1900
https://doi.org/10.3150/18-BEJ1039

# Asymptotically efficient estimators for self-similar stationary Gaussian noises under high frequency observations

MASAAKI FUKASAWA* and TETSUYA TAKABATAKE**

*Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama, Toyonaka, 560-8531, Osaka, Japan. E-mail: *fukasawa@sigmath.es.osaka-u.ac.jp; **takabatake@sigmath.es.osaka-u.ac.jp*

This paper proposes feasible asymptotically efficient estimators for a certain class of Gaussian noises with self-similarity and stationarity properties, which includes the fractional Gaussian noises, under high frequency observations. In this setting, the optimal rate of estimation depends on whether either the Hurst or diffusion parameters is known or not. This is due to the singularity of the asymptotic Fisher information matrix for simultaneous estimation of the above two parameters. One of our key ideas is to extend the Whittle estimation method to the situation of high frequency observations. We show that our estimators are asymptotically efficient in Fisher's sense. Further by Monte-Carlo experiments, we examine finite sample performances of our estimators. Finite sample modifications of the asymptotic variances of the estimators are also given, which exhibit almost perfect fits to the numerical results.

*Keywords:* asymptotic efficiency; fractional Gaussian noises; high frequency observations; local asymptotic normality; Whittle estimation

## References

[1] Bouchaud, J.-P. and Georges, A. (1990). Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. *Phys*. *Rep*. **195** 127–293. MR1081295

[2] Brouste, A. and Fukasawa, M. (2018). Local asymptotic normality property for fractional Gaussian noise under high frequency observations. *Ann*. *Statist*. To appear.

[3] Brouste, A. and Iacus, S.M. (2013). Parameter estimation for the discretely observed fractional Ornstein–Uhlenbeck process and the Yuima R package. *Comput*. *Statist*. **28** 1529–1547. MR3120827

[4] Coeurjolly, J.-F. (2001). Estimating the parameters of a fractional Brownian motion by discrete variations of its sample paths. *Stat*. *Inference Stoch*. *Process*. **4** 199–227. MR1856174

[5] Cohen, S., Gamboa, F., Lacaux, C. and Loubes, J.-M. (2013). LAN property for some fractional type Brownian motion. *ALEA Lat*. *Am*. *J*. *Probab*. *Math*. *Stat*. **10** 91–106. MR3083920

[6] Dahlhaus, R. (1989). Efficient parameter estimation for self-similar processes. *Ann*. *Statist*. **17** 1749–1766. MR1026311

[7] Dahlhaus, R. (2006). Correction: "Efficient parameter estimation for self-similar processes" [Ann. Statist. **17** (1989), no. 4, 1749–1766; MR1026311]. *Ann*. *Statist*. **34** 1045–1047. MR2283403

[8] Dzhaparidze, K. (1986). *Parameter Estimation and Hypothesis Testing in Spectral Analysis of Stationary Time Series*. New York: Springer. MR0812272

[9] Fox, R. and Taqqu, M.S. (1986). Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series. *Ann*. *Statist*. **14** 517–532. MR0840512

[10] Fukasawa, M. and Takabatake, T. (2019). Supplement to "Asymptotically efficient estimators for self-similar stationary Gaussian noises under high frequency observations." DOI:10.3150/18-BEJ1039SUPP.
[11] Ibragimov, I.A. and Has'minskiĭ, R.Z. (1981). *Statistical Estimation*: *Asymptotic Theory*. *Applications of Mathematics* **16**. Berlin: Springer. MR0620321
[12] Kawai, R. (2013). Fisher information for fractional Brownian motion under high-frequency discrete sampling. *Comm*. *Statist*. *Theory Methods* **42** 1628–1636. MR3041490
[13] Kolmogoroff, A.N. (1940). Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *C*. *R*. (*Doklady*) *Acad*. *Sci*. *URSS* (*N*.*S*.) **26** 115–118. MR0003441
[14] Lieberman, O., Rosemarin, R. and Rousseau, J. (2009). Asymptotic theory for maximum likelihood estimation in stationary fractional Gaussian processes, under short-, long- and intermediate memory. *CiteSeerX*.
[15] Lieberman, O., Rosemarin, R. and Rousseau, J. (2012). Asymptotic theory for maximum likelihood estimation of the memory parameter in stationary Gaussian processes. *Econometric Theory* **28** 457–470. MR2913638
[16] Mandelbrot, B.B. and Van Ness, J.W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **10** 422–437. MR0242239
[17] Paxson, V. (1997). Fast, approximate synthesis of fractional Gaussian noise for generating self-similar network traffic. *ACM SIGCOMM Computer Communication Review* **27** 5–18.
[18] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes*: *Stochastic Models with Infinite Variance*. New York: Chapman & Hall. MR1280932
[19] Sinai, Y.G. (1976). Self-similar probability distributions. *Theory Probab*. *Appl*. **21** 64–80.
[20] Taniguchi, M. (1983). On the second order asymptotic efficiency of estimators of Gaussian ARMA processes. *Ann*. *Statist*. **11** 157–169. MR0684873

*Bernoulli* **25**(3), 2019, 1901–1938
https://doi.org/10.3150/18-BEJ1040A

# Sparse covariance matrix estimation in high-dimensional deconvolution

DENIS BELOMESTNY[1,2], MATHIAS TRABS[3] and ALEXANDRE B. TSYBAKOV[4]

[1] *Faculty of Mathematics, Duisburg-Essen University, Thea-Leymann-Str. 9, D-45127 Essen, Germany*
[2] *Higher School of Economics, National Research University, Shabolovka, 26, 119049 Moscow, Russia.*
*E-mail: denis.belomestny@uni-due.de*
[3] *Faculty of Mathematics, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany.*
*E-mail: mathias.trabs@uni-hamburg.de*
[4] *CREST, ENSAE, 5, avenue Henry Le Chatelier, 91120 Palaiseau, France.*
*E-mail: alexandre.tsybakov@ensae.fr*

We study the estimation of the covariance matrix $\Sigma$ of a $p$-dimensional normal random vector based on $n$ independent observations corrupted by additive noise. Only a general nonparametric assumption is imposed on the distribution of the noise without any sparsity constraint on its covariance matrix. In this high-dimensional semiparametric deconvolution problem, we propose spectral thresholding estimators that are adaptive to the sparsity of $\Sigma$. We establish an oracle inequality for these estimators under model miss-specification and derive non-asymptotic minimax convergence rates that are shown to be logarithmic in $n/\log p$. We also discuss the estimation of low-rank matrices based on indirect observations as well as the generalization to elliptical distributions. The finite sample performance of the threshold estimators is illustrated in a numerical example.

*Keywords:* Fourier methods; minimax convergence rates; severely ill-posed inverse problem; thresholding

## References

[1] Belomestny, D. and Reiß, M. (2006). Spectral calibration of exponential Lévy models. *Finance Stoch.* **10** 449–474. MR2276314

[2] Belomestny, D. and Trabs, M. (2017). Low-rank diffusion matrix estimation for high-dimensional time-changed Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.* To appear. Available at arXiv:1510.04638.

[3] Belomestny, D., Trabs, M. and Tsybakov, A.B. (2019). Supplement to "Sparse covariance matrix estimation in high-dimensional deconvolution": A bound for weighted $L^2$-distances of certain densities. DOI:10.3150/18-BEJ1040ASUPP.

[4] Bickel, P.J. and Levina, E. (2008). Covariance regularization by thresholding. *Ann. Statist.* **36** 2577–2604. MR2485008

[5] Bickel, P.J. and Levina, E. (2008). Regularized estimation of large covariance matrices. *Ann. Statist.* **36** 199–227. MR2387969

[6] Butucea, C. and Matias, C. (2005). Minimax estimation of the noise level and of the deconvolution density in a semiparametric convolution model. *Bernoulli* **11** 309–340. MR2132729

[7] Butucea, C., Matias, C. and Pouet, C. (2008). Adaptivity in convolution models with partially known noise distribution. *Electron. J. Stat.* **2** 897–915. MR2447344

1350-7265 

[8] Butucea, C. and Tsybakov, A.B. (2008). Sharp optimality in density deconvolution with dominating bias. I. *Theory Probab. Appl.* **52** 24–39.
[9] Cai, T. and Liu, W. (2011). Adaptive thresholding for sparse covariance matrix estimation. *J. Amer. Statist. Assoc.* **106** 672–684. MR2847949
[10] Cai, T.T., Ren, Z. and Zhou, H.H. (2016). Estimating structured high-dimensional covariance and precision matrices: Optimal rates and adaptive estimation. *Electron. J. Stat.* **10** 1–59. MR3466172
[11] Cai, T.T. and Zhang, A. (2016). Minimax rate-optimal estimation of high-dimensional covariance matrices with incomplete data. *J. Multivariate Anal.* **150** 55–74. MR3534902
[12] Cai, T.T., Zhang, C.-H. and Zhou, H.H. (2010). Optimal rates of convergence for covariance matrix estimation. *Ann. Statist.* **38** 2118–2144. MR2676885
[13] Cai, T.T. and Zhou, H.H. (2012). Minimax estimation of large covariance matrices under $\ell_1$-norm. *Statist. Sinica* **22** 1319–1349. MR3027084
[14] Carroll, R.J. and Hall, P. (1988). Optimal rates of convergence for deconvolving a density. *J. Amer. Statist. Assoc.* **83** 1184–1186. MR0997599
[15] Comte, F. and Lacour, C. (2011). Data-driven density estimation in the presence of additive noise with unknown distribution. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **73** 601–627. MR2853732
[16] Cressie, N. and Wikle, C.K. (2011). *Statistics for Spatio-Temporal Data. Wiley Series in Probability and Statistics*. Hoboken, NJ: Wiley. MR2848400
[17] Dattner, I., Reiß, M. and Trabs, M. (2016). Adaptive quantile estimation in deconvolution with unknown error distribution. *Bernoulli* **22** 143–192. MR3449779
[18] Delaigle, A. and Hall, P. (2016). Methodology for non-parametric deconvolution when the error distribution is unknown. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **78** 231–252. MR3453654
[19] Delaigle, A., Hall, P. and Meister, A. (2008). On deconvolution with repeated measurements. *Ann. Statist.* **36** 665–685. MR2396811
[20] Delaigle, A. and Meister, A. (2011). Nonparametric function estimation under Fourier-oscillating noise. *Statist. Sinica* **21** 1065–1092. MR2827515
[21] Eckle, K., Bissantz, N. and Dette, H. (2017). Multiscale inference for multivariate deconvolution. *Electron. J. Stat.* **11** 4179–4219. MR3716498
[22] El Karoui, N. (2008). Operator norm consistent estimation of large-dimensional sparse covariance matrices. *Ann. Statist.* **36** 2717–2756. MR2485011
[23] Fan, J. (1991). On the optimal rates of convergence for nonparametric deconvolution problems. *Ann. Statist.* **19** 1257–1272. MR1126324
[24] Fan, J., Li, Y. and Yu, K. (2012). Vast volatility matrix estimation using high-frequency data for portfolio selection. *J. Amer. Statist. Assoc.* **107** 412–428. MR2949370
[25] Fan, J., Liao, Y. and Liu, H. (2016). An overview of the estimation of large covariance and precision matrices. *Econom. J.* **19** C1–C32. MR3501529
[26] Fan, J., Liao, Y. and Mincheva, M. (2011). High-dimensional covariance matrix estimation in approximate factor models. *Ann. Statist.* **39** 3320–3356. MR3012410
[27] Fan, J., Liao, Y. and Mincheva, M. (2013). Large covariance estimation by thresholding principal orthogonal complements. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **75** 603–680. With 33 discussions by 57 authors and a reply by Fan, Liao and Mincheva. MR3091653
[28] Fang, K.T., Kotz, S. and Ng, K.W. (1990). *Symmetric Multivariate and Related Distributions. Monographs on Statistics and Applied Probability* **36**. London: Chapman & Hall. MR1071174
[29] Friedman, J., Hastie, T. and Tibshirani, R. (2008). Sparse inverse covariance estimation with the graphical lasso. *Biostatistics* **9** 432–441.
[30] Giné, E. and Nickl, R. (2016). *Mathematical Foundations of Infinite-Dimensional Statistical Models. Cambridge Series in Statistical and Probabilistic Mathematics*. New York: Cambridge Univ. Press. MR3588285

[31] Jacod, J. and Reiss, M. (2014). A remark on the rates of convergence for integrated volatility estimation in the presence of jumps. *Ann. Statist.* **42** 1131–1144. MR3224283

[32] Johannes, J. (2009). Deconvolution with unknown error distribution. *Ann. Statist.* **37** 2301–2323. MR2543693

[33] Kappus, J. and Mabon, G. (2014). Adaptive density estimation in deconvolution problems with unknown error distribution. *Electron. J. Stat.* **8** 2879–2904. MR3299125

[34] Koltchinskii, V., Lounici, K. and Tsybakov, A.B. (2011). Nuclear-norm penalization and optimal rates for noisy low-rank matrix completion. *Ann. Statist.* **39** 2302–2329. MR2906869

[35] Lam, C. and Fan, J. (2009). Sparsistency and rates of convergence in large covariance matrix estimation. *Ann. Statist.* **37** 4254–4278. MR2572459

[36] Lepski, O. and Willer, T. (2017). Estimation in the convolution structure density model. Part I: Oracle inequalities. Preprint. Available at arXiv:1704.04418.

[37] Lepski, O. and Willer, T. (2017). Estimation in the convolution structure density model. Part II: Adaptation over the scale of anisotropic classes. Preprint. Available at arXiv:1704.04420.

[38] Lounici, K. (2014). High-dimensional covariance matrix estimation with missing observations. *Bernoulli* **20** 1029–1058. MR3217437

[39] Low, M.G. (1997). On nonparametric confidence intervals. *Ann. Statist.* **25** 2547–2554. MR1604412

[40] Masry, E. (1993). Strong consistency and rates for deconvolution of multivariate densities of stationary processes. *Stochastic Process. Appl.* **47** 53–74. MR1232852

[41] Matias, C. (2002). Semiparametric deconvolution with unknown noise variance. *ESAIM Probab. Stat.* **6** 271–292. MR1943151

[42] Meister, A. (2008). Deconvolution from Fourier-oscillating error densities under decay and smoothness restrictions. *Inverse Probl.* **24** 015003, 14. MR2384762

[43] Neumann, M.H. (1997). On the effect of estimating the error density in nonparametric deconvolution. *J. Nonparametr. Stat.* **7** 307–330. MR1460203

[44] Rigollet, P. and Tsybakov, A. (2011). Exponential screening and optimal rates of sparse estimation. *Ann. Statist.* **39** 731–771. MR2816337

[45] Rigollet, P. and Tsybakov, A.B. (2012). Comment: "Minimax estimation of large covariance matrices under $\ell_1$-norm" [MR3027084]. *Statist. Sinica* **22** 1358–1367. MR3027087

[46] Rothman, A.J. (2012). Positive definite estimators of large covariance matrices. *Biometrika* **99** 733–740. MR2966781

[47] Rothman, A.J., Levina, E. and Zhu, J. (2009). Generalized thresholding of large covariance matrices. *J. Amer. Statist. Assoc.* **104** 177–186. MR2504372

[48] Sanandaji, B.M., Tascikaraoglu, A., Poolla, K. and Varaiya, P. (2015). Low-dimensional models in spatio-temporal wind speed forecasting. In *American Control Conference* (*ACC*) 4485–4490. IEEE.

[49] Sato, K. (2013). *Lévy Processes and Infinitely Divisible Distributions*. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge: Cambridge Univ. Press. Translated from the 1990 Japanese original, Revised edition of the 1999 English translation. MR3185174

[50] Tao, M., Wang, Y. and Zhou, H.H. (2013). Optimal sparse volatility matrix estimation for high-dimensional Itô processes with measurement errors. *Ann. Statist.* **41** 1816–1864. MR3127850

[51] Tsybakov, A.B. (2009). *Introduction to Nonparametric Estimation*. *Springer Series in Statistics*. New York: Springer. Revised and extended from the 2004 French original, Translated by Vladimir Zaiats. MR2724359

[52] Tsybakov, A.B. (2013). Aggregation and high-dimensional statistics. Saint Flour lecture notes.

*Bernoulli* **25**(3), 2019, 1939–1976
https://doi.org/10.3150/18-BEJ1041

# Hybrid regularisation and the (in)admissibility of ridge regression in infinite dimensional Hilbert spaces

ANIRVAN CHAKRABORTY* and VICTOR M. PANARETOS**

*Institut de Mathématiques, École Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland. E-mail: *anirvan.chakraborty@epfl.ch; **victor.panaretos@epfl.ch*

We consider the problem of estimating the slope function in a functional regression with a scalar response and a functional covariate. This central problem of functional data analysis is well known to be ill-posed, thus requiring a regularised estimation procedure. The two most commonly used approaches are based on spectral truncation or Tikhonov regularisation of the empirical covariance operator. In principle, Tikhonov regularisation is the more canonical choice. Compared to spectral truncation, it is robust to eigenvalue ties, while it attains the optimal minimax rate of convergence in the mean squared sense, and not just in a concentration probability sense. In this paper, we show that, surprisingly, one can strictly improve upon the performance of the Tikhonov estimator in finite samples by means of a linear estimator, while retaining its stability and asymptotic properties by combining it with a form of spectral truncation. Specifically, we construct an estimator that additively decomposes the functional covariate by projecting it onto two orthogonal subspaces defined via functional PCA; it then applies Tikhonov regularisation to the one component, while leaving the other component unregularised. We prove that when the covariate is Gaussian, this hybrid estimator uniformly improves upon the MSE of the Tikhonov estimator in a non-asymptotic sense, effectively rendering it inadmissible. This domination is shown to also persist under discrete observation of the covariate function. The hybrid estimator is linear, straightforward to construct in practice, and with no computational overhead relative to the standard regularisation methods. By means of simulation, it is shown to furnish sizeable gains even for modest sample sizes.

*Keywords:* admissibility; condition index; functional data analysis; ill-posed problem; mean integrated squared error; principal component analysis; rate of convergence; ridge regression; spectral truncation; Tikhonov regularisation

## References

[1] Alquier, P., Gautier, E. and Stoltz, G., eds. (2011). *Inverse Problems and High-Dimensional Estimation. Lecture Notes in Statistics—Proceedings* **203**. Heidelberg: Springer. Lecture notes from the "Stats in the Château" Summer School held in Jouy-en-Josas, August 31–September 4, 2009. MR2867623

[2] Amini, A.A. and Wainwright, M.J. (2012). Sampled forms of functional PCA in reproducing kernel Hilbert spaces. *Ann. Statist.* **40** 2483–2510. MR3097610

[3] Bai, Z. and Saranadasa, H. (1996). Effect of high dimension: By an example of a two sample problem. *Statist. Sinica* **6** 311–329. MR1399305

[4] Cai, T.T. and Hall, P. (2006). Prediction in functional linear regression. *Ann. Statist.* **34** 2159–2179. MR2291496

1350-7265 

[5] Cai, T.T. and Yuan, M. (2011). Optimal estimation of the mean function based on discretely sampled functional data: Phase transition. *Ann. Statist.* **39** 2330–2355. MR2906870

[6] Cardot, H., Ferraty, F. and Sarda, P. (2003). Spline estimators for the functional linear model. *Statist. Sinica* **13** 571–591. MR1997162

[7] Cardot, H. and Johannes, J. (2010). Thresholding projection estimators in functional linear models. *J. Multivariate Anal.* **101** 395–408. MR2564349

[8] Cardot, H., Mas, A. and Sarda, P. (2007). CLT in functional linear regression models. *Probab. Theory Related Fields* **138** 325–361. MR2299711

[9] Cardot, H. and Sarda, P. (2006). Linear regression models for functional data. In *The Art of Semiparametrics*. *Contrib. Statist.* 49–66. Heidelberg: Physica-Verlag/Springer. MR2234875

[10] Chakraborty, A. and Panaretos, V.M. (2019). Supplement to "Hybrid regularisation and the (in)admissibility of ridge regression in infinite dimensional Hilbert spaces". DOI:10.3150/18-BEJ1041SUPP.

[11] Chen, S.X. and Qin, Y.-L. (2010). A two-sample test for high-dimensional data with applications to gene-set testing. *Ann. Statist.* **38** 808–835. MR2604697

[12] Comte, F. and Johannes, J. (2012). Adaptive functional linear regression. *Ann. Statist.* **40** 2765–2797. MR3097959

[13] Conway, J.B. (1978). *Functions of One Complex Variable*, 2nd ed. *Graduate Texts in Mathematics* **11**. New York: Springer. MR0503901

[14] Crambes, C., Kneip, A. and Sarda, P. (2009). Smoothing splines estimators for functional linear regression. *Ann. Statist.* **37** 35–72. MR2488344

[15] Cuevas, A., Febrero, M. and Fraiman, R. (2002). Linear functional regression: The case of fixed design and functional response. *Canad. J. Statist.* **30** 285–300. MR1926066

[16] Ferraty, F. and Vieu, P. (2000). Dimension fractale et estimation de la régression dans des espaces vectoriels semi-normés. *C. R. Acad. Sci. Paris Sér. I Math.* **330** 139–142. MR1745172

[17] Grenander, U. (1981). *Abstract Inference*. *Wiley Series in Probability and Mathematical Statistics*. New York: Wiley. MR0599175

[18] Hall, P. and Horowitz, J.L. (2007). Methodology and convergence rates for functional linear regression. *Ann. Statist.* **35** 70–91. MR2332269

[19] Hall, P. and Hosseini-Nasab, M. (2006). On properties of functional principal components analysis. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **68** 109–126. MR2212577

[20] Hocking, R.R. (2003). *Methods and Applications of Linear Models*: *Regression and the Analysis of Variance*, 2nd ed. *Wiley Series in Probability and Statistics*. Hoboken, NJ: Wiley. MR1963885

[21] Hoerl, A.E. and Kennard, R.W. (1970). Ridge regression: Biased estimation for nonorthogonal problems. *Technometrics* **12** 55–67. DOI:10.2307/1267351.

[22] Horváth, L. and Kokoszka, P. (2012). *Inference for Functional Data with Applications*. *Springer Series in Statistics*. New York: Springer. MR2920735

[23] Hsing, T. and Eubank, R. (2015). *Theoretical Foundations of Functional Data Analysis*, *with an Introduction to Linear Operators*. *Wiley Series in Probability and Statistics*. Chichester: Wiley. MR3379106

[24] Jolliffe, I.T. (2002). *Principal Component Analysis*, 2nd ed. *Springer Series in Statistics*. New York: Springer. MR2036084

[25] Li, Y. and Hsing, T. (2007). On rates of convergence in functional linear regression. *J. Multivariate Anal.* **98** 1782–1804. MR2392433

[26] Lukas, M.A. (1993). Asymptotic optimality of generalized cross-validation for choosing the regularization parameter. *Numer. Math.* **66** 41–66. MR1240702

[27] Lukas, M.A. (2006). Robust generalized cross-validation for choosing the regularization parameter. *Inverse Probl.* **22** 1883–1902. MR2261272

[28] Marx, B.D. and Eilers, P.H. (1999). Generalized linear regression on sampled signals and curves: A P-spline approach. *Technometrics* **41** 1–13.
[29] Meister, A. (2009). *Deconvolution Problems in Nonparametric Statistics*. *Lecture Notes in Statistics* **193**. Berlin: Springer. MR2768576
[30] Ramsay, J.O. and Dalzell, C.J. (1991). Some tools for functional data analysis. *J. Roy. Statist. Soc. Ser. B* **53** 539–572. With discussion and a reply by the authors. MR1125714
[31] Ramsay, J.O. and Silverman, B.W. (2005). *Functional Data Analysis*, 2nd ed. *Springer Series in Statistics*. New York: Springer. MR2168993
[32] Shao, P.Y.-S. and Strawderman, W.E. (1994). Improving on the James–Stein positive-part estimator. *Ann. Statist.* **22** 1517–1538. MR1311987
[33] Theobald, C.M. (1974). Generalizations of mean square error applied to ridge regression. *J. Roy. Statist. Soc. Ser. B* **36** 103–106. MR0368328
[34] Tikhonov, A.N. and Arsenin, V.Y. (1977). *Solutions of Ill-Posed Problems*. *Scripta Series in Mathematics*. Washington, DC: Wiley, New York: V. H. Winston & Sons. Translated from the Russian; preface by translation editor Fritz John. MR0455365
[35] Utreras, F.I. (1987). On generalized cross-validation for multivariate smoothing spline functions. *SIAM J. Sci. Statist. Comput.* **8** 630–643. MR0892310
[36] Wahba, G. (1990). *Spline Models for Observational Data*. *CBMS–NSF Regional Conference Series in Applied Mathematics* **59**. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM). MR1045442
[37] Yao, F., Müller, H.-G. and Wang, J.-L. (2005). Functional linear regression analysis for longitudinal data. *Ann. Statist.* **33** 2873–2903. MR2253106
[38] Yuan, M. and Cai, T.T. (2010). A reproducing kernel Hilbert space approach to functional linear regression. *Ann. Statist.* **38** 3412–3444. MR2766857

*Bernoulli* **25**(3), 2019, 1977–1998
https://doi.org/10.3150/18-BEJ1042

# Consistency of adaptive importance sampling and recycling schemes

JEAN-MICHEL MARIN[1], PIERRE PUDLO[2] and MOHAMMED SEDKI[3]

[1] *IMAG, Univ Montpellier, CNRS, Montpellier, France. E-mail: jean-michel.marin@umontpellier.fr*
[2] *Aix Marseille Université, CNRS, Centrale Marseille, I2M UMR 7373, 13453, Marseille CEDEX 13, France. E-mail: pierre.pudlo@univ-amu.fr*
[3] *Université Paris-Sud, U1181, Hôpital Paul Brousse, Bât. Inserm 15/16, 16 Avenue Paul Vaillant-Couturier 94807 Villejuif CEDEX, France. E-mail: mohammed.sedki@u-psud.fr*

Among Monte Carlo techniques, the importance sampling requires fine tuning of a proposal distribution, which is now fluently resolved through iterative schemes. Sequential adaptive algorithms have been proposed to calibrate the sampling distribution. Cornuet et al. [*Scand. J. Stat.* **39** (2012) 798–812] provides a significant improvement in stability and effective sample size by the introduction of a recycling procedure. However, the consistency of such algorithms have been rarely tackled because of their complexity. Moreover, the recycling strategy of the AMIS estimator adds another difficulty and its consistency remains largely open. In this work, we prove the convergence of sequential adaptive sampling, with finite Monte Carlo sample size at each iteration, and consistency of recycling procedures. Contrary to Douc et al. [*Ann. Statist.* **35** (2007) 420–448], results are obtained here in the asymptotic regime where the number of iterations is going to infinity while the number of drawings per iteration is a fixed, but growing sequence of integers. Hence, some of the results shed new light on adaptive population Monte Carlo algorithms in that last regime and give advices on how the sample sizes should be fixed.

*Keywords:* adaptive algorithms; importance sampling; Monte Carlo methods; population Monte Carlo; sequential Monte Carlo; triangular arrays

## References

[1] Andrieu, C., Doucet, A. and Holenstein, R. (2010). Particle Markov chain Monte Carlo methods. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **72** 269–342. MR2758115

[2] Billingsley, P. (1995). *Probability and Measure*, 3rd ed. *Wiley Series in Probability and Mathematical Statistics*. New York: Wiley. MR1324786

[3] Bugallo, M.F., Martino, L. and Corander, J. (2015). Adaptive importance sampling in signal processing. *Digit. Signal Process.* **47** 36–49. MR3425316

[4] Cameron, E. and Pettitt, A. (2014). Recursive pathways to marginal likelihood estimation with prior-sensitivity analysis. *Statist. Sci.* **29** 397–419. MR3264552

[5] Cappé, O., Guillin, A., Marin, J.M. and Robert, C.P. (2004). Population Monte Carlo. *J. Comput. Graph. Statist.* **13** 907–929. MR2109057

[6] Cappé, O., Guillin, A., Marin, J.-M. and Robert, C.P. (2008). Adaptive importance sampling in general mixture classes. *Stat. Comput.* **18** 587–600.

[7] Cornuet, J.-M., Marin, J.-M., Mira, A. and Robert, C.P. (2012). Adaptive multiple importance sampling. *Scand. J. Stat.* **39** 798–812. MR3000850

[8] Del Moral, P., Doucet, A. and Jasra, A. (2006). Sequential Monte Carlo samplers. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **68** 411–436. MR2278333
[9] Douc, R., Guillin, A., Marin, J.-M. and Robert, C.P. (2007). Convergence of adaptive mixtures of importance sampling schemes. *Ann. Statist.* **35** 420–448. MR2332281
[10] Douc, R., Guillin, A., Marin, J.-M. and Robert, C.P. (2007). Minimum variance importance sampling via population Monte Carlo. *ESAIM Probab. Stat.* **11** 427–447. MR2339302
[11] Feroz, F., Hobson, M., Cameron, E. and Pettitt, A. (2013). Importance nested sampling and the MultiNest algorithm. Preprint. Available at arXiv:1306.2144.
[12] Forbes, F. and Fort, G. (2007). Combining Monte Carlo and mean-field-like methods for inference in hidden Markov random fields. *IEEE Trans. Image Process.* **16** 824–837. MR2460196
[13] He, H.Y. and Owen, A.B. (2014). Optimal mixture weights in multiple importance sampling. Preprint. Available at arXiv:1411.3954.
[14] Hesterberg, T. (1988). Advances in importance sampling. Ph.D. thesis, Stanford University.
[15] Hesterberg, T. (1995). Weighted average importance sampling and defensive mixture distributions. *Technometrics* **37** 185–194.
[16] Liu, J.S. (2008). *Monte Carlo Strategies in Scientific Computing*. *Springer Series in Statistics*. New York: Springer. MR2401592
[17] Martino, L., Elvira, V., Luengo, D. and Corander, J. (2015). An adaptive population importance sampler: Learning from uncertainty. *IEEE Trans. Signal Process.* **63** 4422–4437. MR3368395
[18] Martino, L., Elvira, V., Luengo, D. and Corander, J. (2017). Layered adaptive importance sampling. *Stat. Comput.* **27** 599–623. MR3613588
[19] McLachlan, G.J. and Krishnan, T. (2007). *The EM Algorithm and Extensions*. New York: Wiley.
[20] Owen, A. and Zhou, Y. (2000). Safe and effective importance sampling. *J. Amer. Statist. Assoc.* **95** 135–143. MR1803146
[21] Ripley, B.D. (1987). *Stochastic Simulation*. *Wiley Series in Probability and Mathematical Statistics*: *Applied Probability and Statistics*. New York: Wiley. MR0875224
[22] Robert, C.P. and Casella, G. (2004). *Monte Carlo Statistical Methods*, 2nd ed. *Springer Texts in Statistics*. New York: Springer. MR2080278
[23] Schuster, I. (2015). Gradient importance sampling. Preprint. Available at arXiv:1507.05781.
[24] Schuster, I. (2015). Consistency of importance sampling estimates based on dependent sample sets and an application to models with factorizing likelihoods. Preprint. Available at arXiv:1503.00357.
[25] Sirén, J., Marttinen, P. and Corander, J. (2010). Reconstructing population histories from single-nucleotide polymorphism data. *Mol. Biol. Evol.* **28** 673–683.
[26] Šmídl, V. and Hofman, R. (2014). Efficient sequential Monte Carlo sampling for continuous monitoring of a radiation situation. *Technometrics* **56** 514–528. MR3290621
[27] Van der Vaart, A.W. (2000). *Asymptotic Statistics*. Cambridge: Cambridge University Press.
[28] Veach, E. and Guibas, L.J. (1995). Optimally comabining sampling techniques for Monte Carlo rendering. In *SIGGRAPH'*95 *Proceeding* 419–428. Addison-Wesley.
[29] Xiong, X., Šmídl, V. and Filippone, M. (2017). Adaptive multiple importance sampling for Gaussian processes. *J. Stat. Comput. Simul.* **87** 1644–1665. MR3625236

*Bernoulli* **25**(3), 2019, 1999–2028
https://doi.org/10.3150/18-BEJ1043

# On posterior consistency of tail index for Bayesian kernel mixture models

CHENG LI[1], LIZHEN LIN[2] and DAVID B. DUNSON[3]

[1]*Department of Statistics and Applied Probability, National University of Singapore, Singapore 117546, Singapore. E-mail: stalic@nus.edu.sg*
[2]*Department of Applied and Computational Mathematics and Statistics, The University of Notre Dame, Notre Dame, IN 46556, USA. E-mail: lizhen.lin@nd.edu*
[3]*Department of Statistical Science, Duke University, Durham, NC 27708, USA. E-mail: dunson@duke.edu*

Asymptotic theory of tail index estimation has been studied extensively in the frequentist literature on extreme values, but rarely in the Bayesian context. We investigate whether popular Bayesian kernel mixture models are able to support heavy tailed distributions and consistently estimate the tail index. We show that posterior inconsistency in tail index is surprisingly common for both parametric and nonparametric mixture models. We then present a set of sufficient conditions under which posterior consistency in tail index can be achieved, and verify these conditions for Pareto mixture models under general mixing priors.

*Keywords:* heavy tailed distribution; kernel mixture model; normalized random measures; posterior consistency; tail index

## References

[1] Barrios, E., Lijoi, A., Nieto-Barajas, L.E. and Prünster, I. (2013). Modeling with normalized random measure mixture models. *Statist. Sci.* **28** 313–334. MR3135535

[2] Beirlant, J., Goegebeur, Y., Teugels, J. and Segers, J. (2004). *Statistics of Extremes*: *Theory and Applications*. *Wiley Series in Probability and Statistics*. Chichester: Wiley. With contributions from Daniel De Waal and Chris Ferro. MR2108013

[3] Bottolo, L., Consonni, G., Dellaportas, P. and Lijoi, A. (2003). Bayesian analysis of extreme values by mixture modeling. *Extremes* **6** 25–47. MR2021591

[4] Boucheron, S. and Thomas, M. (2015). Tail index estimation, concentration and adaptivity. *Electron. J. Stat.* **9** 2751–2792. MR3435810

[5] Breiman, L. (1965). On some limit theorems similar to the arc-sin law. *Teor. Veroyatn. Primen.* **10** 351–360. MR0184274

[6] Carpentier, A. and Kim, A.K.H. (2015). Adaptive and minimax optimal estimation of the tail coefficient. *Statist. Sinica* **25** 1133–1144. MR3410301

[7] Clauset, A., Shalizi, C.R. and Newman, M.E.J. (2009). Power-law distributions in empirical data. *SIAM Rev.* **51** 661–703. MR2563829

[8] Cormann, U. and Reiss, R.-D. (2009). Generalizing the Pareto to the log-Pareto model and statistical inference. *Extremes* **12** 93–105. MR2480725

[9] de Haan, L. and Resnick, S.I. (1980). A simple asymptotic estimate for the index of a stable distribution. *J. Roy. Statist. Soc. Ser. B* **42** 83–87. MR0567205

[10] Diebolt, J., El-Aroui, M.-A., Garrido, M. and Girard, S. (2005). Quasi-conjugate Bayes estimates for GPD parameters and application to heavy tails modelling. *Extremes* **8** 57–78. MR2201901

1350-7265 

[11] do Nascimento, F.F., Gamerman, D. and Lopes, H.F. (2012). A semiparametric Bayesian approach to extreme value estimation. *Stat. Comput.* **22** 661–675. MR2865043
[12] Doss, H. and Sellke, T. (1982). The tails of probabilities chosen from a Dirichlet prior. *Ann. Statist.* **10** 1302–1305. MR0673666
[13] Drees, H. (1998). Optimal rates of convergence for estimates of the extreme value index. *Ann. Statist.* **26** 434–448. MR1608148
[14] Drees, H. (2001). Minimax risk bounds in extreme value theory. *Ann. Statist.* **29** 266–294. MR1833966
[15] Escobar, M.D. and West, M. (1995). Bayesian density estimation and inference using mixtures. *J. Amer. Statist. Assoc.* **90** 577–588. MR1340510
[16] Favaro, S. and Teh, Y.W. (2013). MCMC for normalized random measure mixture models. *Statist. Sci.* **28** 335–359. MR3135536
[17] Ferguson, T.S. (1973). A Bayesian analysis of some nonparametric problems. *Ann. Statist.* **1** 209–230. MR0350949
[18] Frigessi, A., Haug, O. and Rue, H. (2002). A dynamic mixture model for unsupervised tail estimation without threshold selection. *Extremes* **5** 219–235. MR1995776
[19] Fristedt, B.E. (1967). Sample function behavior of increasing processes with stationary, independent increments. *Pacific J. Math.* **21** 21–33. MR0210190
[20] Fristedt, B.E. and Pruitt, W.E. (1971). Lower functions for increasing random walks and subordinators. *Z. Wahrsch. Verw. Gebiete* **18** 167–182. MR0292163
[21] Fúquene Patiño, J.A. (2015). A semi-parametric Bayesian extreme value model using a Dirichlet process mixture of gamma densities. *J. Appl. Stat.* **42** 267–280. MR3276944
[22] Ghosal, S., Ghosh, J.K. and Ramamoorthi, R.V. (1999). Posterior consistency of Dirichlet mixtures in density estimation. *Ann. Statist.* **27** 143–158. MR1701105
[23] Ghosal, S., Ghosh, J.K. and van der Vaart, A.W. (2000). Convergence rates of posterior distributions. *Ann. Statist.* **28** 500–531. MR1790007
[24] Ghosal, S. and van der Vaart, A. (2007). Posterior convergence rates of Dirichlet mixtures at smooth densities. *Ann. Statist.* **35** 697–723. MR2336864
[25] Ghosh, J.K. and Ramamoorthi, R.V. (2003). *Bayesian Nonparametrics*. *Springer Series in Statistics*. New York: Springer. MR1992245
[26] Green, P.J. and Richardson, S. (2001). Modelling heterogeneity with and without the Dirichlet process. *Scand. J. Stat.* **28** 355–375. MR1842255
[27] Haeusler, E. and Teugels, J.L. (1985). On asymptotic normality of Hill's estimator for the exponent of regular variation. *Ann. Statist.* **13** 743–756. MR0790569
[28] Hall, P. (1982). On some simple estimates of an exponent of regular variation. *J. Roy. Statist. Soc. Ser. B* **44** 37–42. MR0655370
[29] Hall, P. and Welsh, A.H. (1984). Best attainable rates of convergence for estimates of parameters of regular variation. *Ann. Statist.* **12** 1079–1084. MR0751294
[30] Hall, P. and Welsh, A.H. (1985). Adaptive estimates of parameters of regular variation. *Ann. Statist.* **13** 331–341. MR0773171
[31] Hill, B.M. (1975). A simple general approach to inference about the tail of a distribution. *Ann. Statist.* **3** 1163–1174. MR0378204
[32] James, L.F., Lijoi, A. and Prünster, I. (2009). Posterior analysis for normalized random measures with independent increments. *Scand. J. Stat.* **36** 76–97. MR2508332
[33] Kruijer, W., Rousseau, J. and van der Vaart, A. (2010). Adaptive Bayesian density estimation with location-scale mixtures. *Electron. J. Stat.* **4** 1225–1257. MR2735885
[34] Lange, K. (1973). Borel sets of probability measures. *Pacific J. Math.* **48** 141–161. MR0357723

[35] Li, C., Lin, L. and Dunson, D.B. (2019). Supplement to “On posterior consistency of tail index for Bayesian kernel mixture models.” DOI:10.3150/18-BEJ1043SUPP.
[36] Lijoi, A., Mena, R.H. and Prünster, I. (2007). Controlling the reinforcement in Bayesian nonparametric mixture models. *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **69** 715–740. MR2370077
[37] Lijoi, A. and Prünster, I. (2010). Models beyond the Dirichlet process. In *Bayesian Nonparametrics. Camb. Ser. Stat. Probab. Math.* **28** 80–136. Cambridge: Cambridge Univ. Press. MR2730661
[38] Lo, A.Y. (1984). On a class of Bayesian nonparametric estimates. I. Density estimates. *Ann. Statist.* **12** 351–357. MR0733519
[39] MacEachern, S.N. (1994). Estimating normal means with a conjugate style Dirichlet process prior. *Comm. Statist. Simulation Comput.* **23** 727–741. MR1293996
[40] Mason, D.M. (1982). Laws of large numbers for sums of extreme values. *Ann. Probab.* **10** 754–764. MR0659544
[41] Neal, R.M. (2000). Markov chain sampling methods for Dirichlet process mixture models. *J. Comput. Graph. Statist.* **9** 249–265. MR1823804
[42] Novak, S.Y. (2014). Lower bounds to the accuracy of inference on heavy tails. *Bernoulli* **20** 979–989. MR3178524
[43] Pickands, J. III (1975). Statistical inference using extreme order statistics. *Ann. Statist.* **3** 119–131. MR0423667
[44] Regazzini, E., Lijoi, A. and Prünster, I. (2003). Distributional results for means of normalized random measures with independent increments. *Ann. Statist.* **31** 560–585. Dedicated to the memory of Herbert E. Robbins. MR1983542
[45] Richardson, S. and Green, P.J. (1997). On Bayesian analysis of mixtures with an unknown number of components. *J. Roy. Statist. Soc. Ser. B* **59** 731–792. MR1483213
[46] Schwartz, L. (1965). On Bayes procedures. *Z. Wahrsch. Verw. Gebiete* **4** 10–26. MR0184378
[47] Shen, W., Tokdar, S.T. and Ghosal, S. (2013). Adaptive Bayesian multivariate density estimation with Dirichlet mixtures. *Biometrika* **100** 623–640. MR3094441
[48] Shorack, G.R. and Wellner, J.A. (1986). *Empirical Processes with Applications to Statistics. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics*. New York: Wiley. MR0838963
[49] Stephenson, A. and Tawn, J. (2004). Bayesian inference for extremes: Accounting for the three extremal types. *Extremes* **7** 291–307. MR2212389
[50] Tokdar, S.T. (2006). Posterior consistency of Dirichlet location-scale mixture of normals in density estimation and regression. *Sankhyā* **68** 90–110. MR2301566
[51] Tressou, J. (2008). Bayesian nonparametrics for heavy tailed distribution. Application to food risk assessment. *Bayesian Anal.* **3** 367–391. MR2407431
[52] Wang, Z., Rodriguez, A. and Kottas, A. (2012). A nonparametric mixture modeling framework for extreme value analysis. Technical report. https://www.soe.ucsc.edu/research/technical-reports/UCSC-SOE-11-26/download.
[53] Watanabe, T. (1960). A probabilistic method in Hausdorff moment problem and Laplace–Stieltjes transform. *J. Math. Soc. Japan* **12** 192–206. MR0120683
[54] Wu, Y. and Ghosal, S. (2008). Kullback Leibler property of kernel mixture priors in Bayesian density estimation. *Electron. J. Stat.* **2** 298–331. MR2399197

*Bernoulli* **25**(3), 2019, 2029–2050
https://doi.org/10.3150/18-BEJ1044

# The unusual properties of aggregated superpositions of Ornstein–Uhlenbeck type processes

DANIJEL GRAHOVAC[1], NIKOLAI N. LEONENKO[2], ALLA SIKORSKII[3] and MURAD S. TAQQU[4]

[1]*Department of Mathematics, University of Osijek, Trg Ljudevita Gaja 6, 31000 Osijek, Croatia. E-mail: dgrahova@mathos.hr*
[2]*School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, Wales, CF24 4AG, UK. E-mail: LeonenkoN@cardiff.ac.uk*
[3]*Department of Psychiatry and Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA. E-mail: sikorska@stt.msu.edu*
[4]*Department of Mathematics and Statistics, Boston University, Boston, MA 02215, USA. E-mail: murad@bu.edu*

Superpositions of Ornstein–Uhlenbeck type (supOU) processes form a rich class of stationary processes with a flexible dependence structure. The asymptotic behavior of the integrated and partial sum supOU processes can be, however, unusual. Their cumulants and moments turn out to have an unexpected rate of growth. We identify the property of fast growth of moments or cumulants as *intermittency*. Many proofs are given in a supplemental article (Grahovac, Leonenko, Sikorskii and Taqqu (2018)).

*Keywords:* cumulants; intermittency; moments; Ornstein–Uhlenbeck process; self-similarity; supOU processes

## References

[1] Abramowitz, M. and Stegun, I.A., eds. (1992). *Handbook of Mathematical Functions with Formulas*, *Graphs*, *and Mathematical Tables*. New York: Dover Publications, Inc. MR1225604

[2] Barndorff-Nielsen, O.E. (1998). Processes of normal inverse Gaussian type. *Finance Stoch*. **2** 41–68. MR1804664

[3] Barndorff-Nielsen, O.E. (2001). Superposition of Ornstein–Uhlenbeck type processes. *Theory Probab*. *Appl*. **45** 175–194.

[4] Barndorff-Nielsen, O.E. and Leonenko, N.N. (2005). Burgers' turbulence problem with linear or quadratic external potential. *J*. *Appl*. *Probab*. **42** 550–565. MR2145493

[5] Barndorff-Nielsen, O.E. and Leonenko, N.N. (2005). Spectral properties of superpositions of Ornstein–Uhlenbeck type processes. *Methodol*. *Comput*. *Appl*. *Probab*. **7** 335–352. MR2210585

[6] Barndorff-Nielsen, O.E., Pakkanen, M.S. and Schmiegel, J. (2014). Assessing relative volatility/intermittency/energy dissipation. *Electron*. *J*. *Stat*. **8** 1996–2021. MR3273617

[7] Barndorff-Nielsen, O.E., Pérez-Abreu, V. and Thorbjørnsen, S. (2013). Lévy mixing. *ALEA Lat*. *Am*. *J*. *Probab*. *Math*. *Stat*. **10** 1013–1062. MR3151748

1350-7265 

[8] Barndorff-Nielsen, O.E. and Schmiegel, J. (2009). Brownian semistationary processes and volatility/intermittency. In *Advanced Financial Modelling*. *Radon Ser*. *Comput*. *Appl*. *Math*. **8** 1–25. Berlin: Walter de Gruyter. MR2648456

[9] Barndorff-Nielsen, O.E. and Shephard, N. (2001). Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *J*. *R*. *Stat*. *Soc*. *Ser*. *B*. *Stat*. *Methodol*. **63** 167–241. MR1841412

[10] Barndorff-Nielsen, O.E. and Stelzer, R. (2011). Multivariate supOU processes. *Ann*. *Appl*. *Probab*. **21** 140–182. MR2759198

[11] Barndorff-Nielsen, O.E. and Stelzer, R. (2013). The multivariate supOU stochastic volatility model. *Math*. *Finance* **23** 275–296. MR3034078

[12] Barndorff-Nielsen, O.E. and Veraart, A.E. (2013). Stochastic volatility of volatility and variance risk premia. *J*. *Financ*. *Econom*. **11** 1–46.

[13] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation*. *Encyclopedia of Mathematics and Its Applications* **27**. Cambridge: Cambridge Univ. Press. MR0898871

[14] Carmona, R.A. and Molchanov, S.A. (1994). Parabolic Anderson problem and intermittency. *Mem*. *Amer*. *Math*. *Soc*. **108** viii + 125. MR1185878

[15] Chen, L. and Dalang, R.C. (2015). Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Ann*. *Probab*. **43** 3006–3051. MR3433576

[16] Davydov, Ju.A. (1968). The convergence of distributions which are generated by stationary random processes. *Theory Probab*. *Appl*. **13** 691–696.

[17] Embrechts, P. and Maejima, M. (2002). *Selfsimilar Processes*. *Princeton Series in Applied Mathematics*. Princeton, NJ: Princeton Univ. Press. MR1920153

[18] Fasen, V. and Klüppelberg, C. (2007). Extremes of supOU processes. In *Stochastic Analysis and Applications*. *Abel Symp*. **2** 339–359. Berlin: Springer. MR2397794

[19] Frisch, U. (1995). *Turbulence*: *The Legacy of A*. *N*. *Kolmogorov*. Cambridge: Cambridge Univ. Press. MR1428905

[20] Gärtner, J., König, W. and Molchanov, S. (2007). Geometric characterization of intermittency in the parabolic Anderson model. *Ann*. *Probab*. **35** 439–499. MR2308585

[21] Grahovac, D., Leonenko, N.N., Sikorskii, A. and Taqqu, M.S. (2019). Supplement to "The unusual properties of aggregated superpositions of Ornstein–Uhlenbeck type processes." DOI:10.3150/18-BEJ1044SUPP.

[22] Grahovac, D., Leonenko, N.N., Sikorskii, A. and Tešnjak, I. (2016). Intermittency of superpositions of Ornstein–Uhlenbeck type processes. *J*. *Stat*. *Phys*. **165** 390–408. MR3554387

[23] Griffin, J.E. and Steel, M.F.J. (2010). Bayesian inference with stochastic volatility models using continuous superpositions of non-Gaussian Ornstein–Uhlenbeck processes. *Comput*. *Statist*. *Data Anal*. **54** 2594–2608. MR2720463

[24] Heyde, C.C. and Leonenko, N.N. (2005). Student processes. *Adv*. *in Appl*. *Probab*. **37** 342–365. MR2144557

[25] Ivanov, A.V. and Leonenko, N.N. (1989). *Statistical Analysis of Random Fields*. *Mathematics and Its Applications* (*Soviet Series*) **28**. Dordrecht: Kluwer Academic. MR1009786

[26] Jurek, Z.J. (2001). Remarks on the selfdecomposability and new examples. *Demonstratio Math*. **34** 241–250. MR1833180

[27] Khoshnevisan, D. (2014). *Analysis of Stochastic Partial Differential Equations*. *CBMS Regional Conference Series in Mathematics* **119**. Providence, RI: Amer. Math. Soc. MR3222416

[28] Leonenko, N. and Taufer, E. (2005). Convergence of integrated superpositions of Ornstein–Uhlenbeck processes to fractional Brownian motion. *Stochastics* **77** 477–499. MR2190797

[29] Lukacs, E. (1970). *Characteristic Functions*, 2nd edn. New York: Hafner Publishing Co. MR0346874

[30] Masuda, H. (2004). On multidimensional Ornstein–Uhlenbeck processes driven by a general Lévy process. *Bernoulli* **10** 97–120. MR2044595
[31] Molchanov, S.A. (1991). Ideas in the theory of random media. *Acta Appl. Math.* **22** 139–282. MR1111743
[32] Moser, M. and Stelzer, R. (2011). Tail behavior of multivariate Lévy-driven mixed moving average processes and supOU stochastic volatility models. *Adv. in Appl. Probab.* **43** 1109–1135. MR2867948
[33] Oodaira, H. and Yoshihara, K. (1972). Functional central limit theorems for strictly stationary processes satisfying the strong mixing condition. *Kōdai Math. Semin. Rep.* **24** 259–269. MR0317380
[34] Podolskij, M. (2015). Ambit fields: Survey and new challenges. In *XI Symposium on Probability and Stochastic Processes. Progress in Probability* **69** 241–279. Cham: Birkhäuser/Springer. MR3558145
[35] Rajput, B.S. and Rosiński, J. (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Related Fields* **82** 451–487. MR1001524
[36] Stelzer, R., Tosstorff, T. and Wittlinger, M. (2015). Moment based estimation of supOU processes and a related stochastic volatility model. *Stat. Risk Model.* **32** 1–24. MR3331779
[37] Stelzer, R. and Zavišin, J. (2015). Derivative pricing under the possibility of long memory in the supOU stochastic volatility model. In *Innovations in Quantitative Risk Management: TU München, September* 2013 **99** 75–92. Springer.
[38] Stoyanov, J.M. (1997). *Counterexamples in Probability. Wiley Series in Probability and Statistics*. Chichester: Wiley. MR3444842
[39] Taqqu, M.S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrsch. Verw. Gebiete* **31** 287–302. MR0400329
[40] Yokoyama, R. (1980). Moment bounds for stationary mixing sequences. *Z. Wahrsch. Verw. Gebiete* **52** 45–57. MR0568258
[41] Zel'dovich, Ya.B., Molchanov, S.A., Ruzmaĭkin, A.A. and Sokolov, D.D. (1987). Intermittency in random media. *Sov. Phys., Usp.* **30** 353.

*Bernoulli* **25**(3), 2019, 2051–2074
https://doi.org/10.3150/18-BEJ1045

# Gibbs–non-Gibbs transitions in the fuzzy Potts model with a Kac-type interaction: Closing the Ising gap

FLORIAN HENNING*, RICHARD C. KRAAIJ** and CHRISTOF KÜLSKE†,‡

*Ruhr-Universität Bochum, Fakultät für Mathematik, 44801 Bochum, Germany.*
*E-mail:* *Florian.Henning@rub.de; **kraaij.rc@gmail.com; †Christof.Kuelske@ruhr-uni-bochum.de; url: ‡http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Kuelske/kuelske.html

We complete the investigation of the Gibbs properties of the fuzzy Potts model on the $d$-dimensional torus with Kac interaction which was started by Jahnel and one of the authors in (Sharp thresholds for Gibbs-non-Gibbs transitions in the fuzzy Potts model with a Kac-type interaction (2017)). As our main result of the present paper, we extend the previous sharpness result of mean-field bounds to cover all possible cases of fuzzy transformations, allowing also for the occurrence of Ising classes (containing precisely two spin values). The closing of this previously left open Ising-gap involves an analytical argument showing uniqueness of minimizing profiles for certain non-homogeneous conditional variational problems.

*Keywords:* diluted large deviation principle; fuzzy Kac–Potts model; Gibbs versus non-Gibbs; Kac model; large deviation principles; Potts model

## References

[1] Bogachev, V.I. (2007). *Measure Theory. Vol. I, II*. Berlin: Springer. MR2267655
[2] den Hollander, F., Redig, F. and van Zuijlen, W. (2015). Gibbs–non-Gibbs dynamical transitions for mean-field interacting Brownian motions. *Stochastic Process. Appl.* **125** 371–400. MR3274705
[3] Ermolaev, V. and Külske, C. (2010). Low-temperature dynamics of the Curie–Weiss model: Periodic orbits, multiple histories, and loss of Gibbsianness. *J. Stat. Phys.* **141** 727–756. MR2739308
[4] Fernández, R., den Hollander, F. and Martínez, J. (2013). Variational description of Gibbs–non-Gibbs dynamical transitions for the Curie–Weiss model. *Comm. Math. Phys.* **319** 703–730. MR3040373
[5] Fernández, R., den Hollander, F. and Martínez, J. (2014). Variational description of Gibbs–non-Gibbs dynamical transitions for spin-flip systems with a Kac-type interaction. *J. Stat. Phys.* **156** 203–220. MR3215620
[6] Georgii, H.-O. (2011). *Gibbs Measures and Phase Transitions*, 2nd ed. *De Gruyter Studies in Mathematics* **9**. Berlin: de Gruyter. MR2807681
[7] Häggström, O. (2003). Is the fuzzy Potts model Gibbsian? *Ann. Inst. Henri Poincaré Probab. Stat.* **39** 891–917. MR1997217
[8] Häggström, O. and Külske, C. (2004). Gibbs properties of the fuzzy Potts model on trees and in mean field. *Markov Process. Related Fields* **10** 477–506. MR2097868
[9] Jahnel, B. and Külske, C. (2017). Gibbsian representation for point processes via hyperedge potentials. Preprint. Available at arXiv:1707.05991.
[10] Jahnel, B. and Külske, C. (2017). Sharp thresholds for Gibbs–non-Gibbs transitions in the fuzzy Potts model with a Kac-type interaction. *Bernoulli* **23** 2808–2827. MR3648046

1350-7265 

[11] Jahnel, B. and Külske, C. (2017). The Widom–Rowlinson model under spin flip: Immediate loss and sharp recovery of quasilocality. *Ann. Appl. Probab.* **27** 3845–3892. MR3737939

[12] Klenke, A. (2014). *Probability Theory. A Comprehensive Course*, 2nd ed. *Universitext*. London: Springer. MR3112259

[13] Külske, C. (1999). (Non-) Gibbsianness and phase transitions in random lattice spin models. *Markov Process. Related Fields* **5** 357–383. MR1734240

[14] Külske, C. and Le Ny, A. (2007). Spin-flip dynamics of the Curie–Weiss model: Loss of Gibbsianness with possibly broken symmetry. *Comm. Math. Phys.* **271** 431–454. MR2287911

[15] Maes, C. and Vande Velde, K. (1995). The fuzzy Potts model. *J. Phys. A* **28** 4261–4270. MR1351929

[16] Ruelle, D. (1999). *Statistical Mechanics*: *Rigorous Results*. River Edge, NJ: World Scientific. London: Imperial College Press. MR1747792

[17] van Enter, A.C.D., Fernández, R. and Sokal, A.D. (1993). Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory. *J. Stat. Phys.* **72** 879–1167. MR1241537

*Bernoulli* **25**(3), 2019, 2075–2106
https://doi.org/10.3150/18-BEJ1046

# Regularization, sparse recovery, and median-of-means tournaments

GÁBOR LUGOSI[1,2] and SHAHAR MENDELSON[3,4]

[1]*Department of Economics and Business, Pompeu Fabra University, Barcelona, Spain.*
[2]*ICREA, Pg. Lluís Companys 23, 08010 Barcelona, Spain. E-mail: gabor.lugosi@upf.edu*
[3]*Mathematical Sciences Institute, The Australian National University, Canberra, ACT 0200 Australia.*
[4]*LPSM, Sorbonne Université, Paris, France. E-mail: shahar.mendelson@anu.edu.au*

We introduce a regularized risk minimization procedure for regression function estimation. The procedure is based on median-of-means tournaments, introduced by the authors in Lugosi and Mendelson (2018) and achieves near optimal accuracy and confidence under general conditions, including heavy-tailed predictor and response variables. It outperforms standard regularized empirical risk minimization procedures such as LASSO or SLOPE in heavy-tailed problems.

*Keywords:* LASSO; median-of-means tournament; regularized risk minimization; robust regression; SLOPE

## References

[1] Audibert, J.-Y. and Catoni, O. (2011). Robust linear least squares regression. *Ann. Statist.* **39** 2766–2794. MR2906886
[2] Bellec, P., Lecué, G. and Tsybakov, A. (2016). Slope meets lasso: Improved oracle bounds and optimality. Preprint. Available at arXiv:1605.08651.
[3] Brownlees, C., Joly, E. and Lugosi, G. (2015). Empirical risk minimization for heavy-tailed losses. *Ann. Statist.* **43** 2507–2536. MR3405602
[4] Goldenshluger, A. and Nemirovski, A. (1997). On spatially adaptive estimation of nonparametric regression. *Math. Methods Statist.* **6** 135–170. MR1466625
[5] Hsu, D. and Sabato, S. (2013). Approximate loss minimization with heavy tails. *Computing Research Repository*. abs/1307.1827.
[6] Lecué, G. and Lerasle, M. (2017). Learning from MOM's principles. Preprint. Available at arXiv:1701.01961.
[7] Lecué, G. and Mendelson, S. (2017). Sparse recovery under weak moment assumptions. *J. Eur. Math. Soc.* (*JEMS*) **19** 881–904. MR3612870
[8] Lecué, G. and Mendelson, S. (2018). Learning subgaussian classes: Upper and minimax bounds. In *Topics in Learning Theory* (S. Boucheron and N. Vayatis, eds.) Societe Mathematique de France. To appear.
[9] Lecué, G. and Mendelson, S. (2018). Regularization and the small-ball method I: Sparse recovery. *Ann. Statist.* **46** 611–641. MR3782379
[10] Lerasle, M. and Oliveira, R.I. (2012). Robust empirical mean estimators. Unpublished manuscript.
[11] Lugosi, G. and Mendelson, S. (2018). Risk minimization by median-of-means tournaments. *J. Eur. Math. Soc.* (*JEMS*). To appear.
[12] Lugosi, G. and Mendelson, S. (2018). Sub-Gaussian estimators of the mean of a random vector. *Ann. Statist.* To appear.

[13] Mendelson, S. (2015). Learning without concentration. *J. ACM* **62** Art. 21, 25. MR3367000
[14] Mendelson, S. (2017). “Local” vs. “global” parameters – breaking the Gaussian complexity barrier. *Ann. Statist.* **45** 1835–1862. MR3718154
[15] Mendelson, S. (2017). On aggregation for heavy-tailed classes. *Probab. Theory Related Fields* **168** 641–674. MR3663627
[16] Mendelson, S. (2017). On multiplier processes under weak moment assumptions. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math.* **2169** 301–318. Springer, Cham. MR3645129
[17] Mendelson, S. (2017). An optimal unrestricted learning procedure. Preprint. Available at arXiv:1707.05342v2.
[18] Minsker, S. (2015). Geometric median and robust estimation in Banach spaces. *Bernoulli* **21** 2308–2335. MR3378468
[19] Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. *J. Roy. Statist. Soc. Ser. B* **58** 267–288. MR1379242

*Bernoulli* **25**(3), 2019, 2107–2136
https://doi.org/10.3150/18-BEJ1047

# Root-$n$ consistent estimation of the marginal density in semiparametric autoregressive time series models

LIONEL TRUQUET

*ENSAI – Campus de Ker-Lann, Rue Blaise Pascal – BP 37203, 35712 BRUZ cedex, France.*
*E-mail: lionel.truquet@ensai.fr*

In this paper, we consider the problem of estimating the marginal density in some autoregressive time series models for which the conditional mean and variance have a parametric specification. Under some regularity conditions, we show that a kernel type estimate based on the residuals can be root-$n$ consistent even if the noise density is unknown. Our results substantially extend those existing in the literature. Our assumptions are carefully checked for some standard time series models such as ARMA or GARCH processes. Asymptotic expansion of our estimator is obtained by combining some martingale type arguments and a coupling method for time series which is of independent interest. We also study the uniform convergence of our estimator on compact intervals.

*Keywords:* kernel density estimation; nonlinear time series

## References

[1] Aït-Sahalia, Y. (1996). Testing continuous-time models of the spot interest rate. *Rev. Financ. Stud.* **9** 385–426.

[2] Ango Nze, P. and Doukhan, P. (1998). Functional estimation for time series: Uniform convergence properties. *J. Statist. Plann. Inference* **68** 5–29. MR1627588

[3] Bosq, D. (1998). *Nonparametric Statistics for Stochastic Processes: Estimation and Prediction*, 2nd ed. *Lecture Notes in Statistics* **110**. New York: Springer. MR1640691

[4] Brockwell, P.J. and Davis, R.A. (1991). *Time Series: Theory and Methods*, 2nd ed. *Springer Series in Statistics*. New York: Springer. MR1093459

[5] Corradi, V. and Swanson, N.R. (2005). Bootstrap specification tests for diffusion processes. *J. Econometrics* **124** 117–148. MR2067684

[6] Dedecker, J., Doukhan, P., Lang, G., León, J.R., Louhichi, S. and Prieur, C. (2007). *Weak Dependence: With Examples and Applications*. *Lecture Notes in Statistics* **190**. New York: Springer. MR2338725

[7] Delaigle, A., Meister, A. and Rombouts, J. (2016). Root-$T$ consistent density estimation in GARCH models. *J. Econometrics* **192** 55–63. MR3463664

[8] Doukhan, P. and Louhichi, S. (2001). Functional estimation of a density under a new weak dependence condition. *Scand. J. Stat.* **28** 325–341. MR1842253

[9] Francq, C. and Zakoïan, J.-M. (2010). *GARCH Models: Structure, Statistical Inference and Financial Applications*. Chichester: Wiley. MR3186556

[10] Francq, C. and Zakoïan, J.-M. (2013). Estimating the marginal law of a time series with applications to heavy-tailed distributions. *J. Bus. Econom. Statist.* **31** 412–425. MR3173691

1350-7265 

[11] Frees, E.W. (1994). Estimating densities of functions of observations. *J. Amer. Statist. Assoc.* **89** 517–525. MR1294078
[12] Gao, J. and King, M. (2004). Adaptive testing in continuous-time diffusion models. *Econometric Theory* **20** 844–882. MR2089144
[13] Giné, E. and Mason, D.M. (2007). On local $U$-statistic processes and the estimation of densities of functions of several sample variables. *Ann. Statist.* **35** 1105–1145. MR2341700
[14] Kim, K.H., Zhang, T. and Wu, W.B. (2015). Parametric specification test for nonlinear autoregressive models. *Econometric Theory* **31** 1078–1101. MR3396240
[15] Li, S. and Tu, Y. (2016). $\sqrt{n}$-consistent density estimation in semiparametric regression models. *Comput. Statist. Data Anal.* **104** 91–109. MR3540989
[16] Liao, Y. and Stachurski, J. (2015). Simulation-based density estimation for time series using covariate data. *J. Bus. Econom. Statist.* **33** 595–606. MR3416603
[17] Liu, W. and Wu, W.B. (2010). Simultaneous nonparametric inference of time series. *Ann. Statist.* **38** 2388–2421. MR2676893
[18] Müller, U.U. (2012). Estimating the density of a possibly missing response variable in nonlinear regression. *J. Statist. Plann. Inference* **142** 1198–1214. MR2879764
[19] Robinson, P.M. (1983). Nonparametric estimators for time series. *J. Time Series Anal.* **4** 185–207. MR0732897
[20] Roussas, G.G. (2000). Asymptotic normality of the kernel estimate of a probability density function under association. *Statist. Probab. Lett.* **50** 1–12. MR1804620
[21] Saavedra, Á. and Cao, R. (1999). Rate of convergence of a convolution-type estimator of the marginal density of an MA(1) process. *Stochastic Process. Appl.* **80** 129–155. MR1682259
[22] Schick, A. and Wefelmeyer, W. (2004). Root $n$ consistent and optimal density estimators for moving average processes. *Scand. J. Stat.* **31** 63–78. MR2042599
[23] Schick, A. and Wefelmeyer, W. (2004). Root $n$ consistent density estimators for sums of independent random variables. *J. Nonparametr. Stat.* **16** 925–935. MR2094747
[24] Schick, A. and Wefelmeyer, W. (2007). Uniformly root-$N$ consistent density estimators for weakly dependent invertible linear processes. *Ann. Statist.* **35** 815–843. MR2336870
[25] Schick, A. and Wefelmeyer, W. (2009). Convergence rates of density estimators for sums of powers of observations. *Metrika* **69** 249–264. MR2481923
[26] Tong, H. (1990). *Nonlinear Time Series: A Dynamical System Approach. Oxford Statistical Science Series* **6**. New York: The Clarendon Press, Oxford Univ. Press. MR1079320
[27] Truquet, L. (2019). Supplement to "Root-$n$ consistent estimation of the marginal density in semiparametric autoregressive time series models." DOI:10.3150/18-BEJ1047SUPP.
[28] van der Vaart, A.W. (1998). *Asymptotic Statistics. Cambridge Series in Statistical and Probabilistic Mathematics* **3**. Cambridge: Cambridge Univ. Press. MR1652247
[29] Zhao, Z. (2010). Density estimation for nonlinear parametric models with conditional heteroscedasticity. *J. Econometrics* **155** 71–82. MR2592850

*Bernoulli* **25**(3), 2019, 2137–2162
https://doi.org/10.3150/18-BEJ1048

# Integration with respect to the non-commutative fractional Brownian motion

AURÉLIEN DEYA* and RENÉ SCHOTT**

*Institut Elie Cartan, University of Lorraine B.P. 239, 54506 Vandoeuvre-lès-Nancy, Cedex France. E-mail: *aurelien.deya@univ-lorraine.fr; **rene.schott@univ-lorraine.fr*

We study the issue of integration with respect to the non-commutative fractional Brownian motion, that is the analog of the standard fractional Brownian motion in a non-commutative probability setting.

When the Hurst index $H$ of the process is stricly larger than $1/2$, integration can be handled through the so-called Young procedure. The situation where $H = 1/2$ corresponds to the specific free case, for which an Itô-type approach is known to be possible.

When $H < 1/2$, rough-path-type techniques must come into the picture, which, from a theoretical point of view, involves the use of some a-priori-defined Lévy area process. We show that such an object can indeed be "canonically" constructed for any $H \in (\frac{1}{4}, \frac{1}{2})$. Finally, when $H \leq 1/4$, we exhibit a similar non-convergence phenomenon as for the non-diagonal entries of the (classical) Lévy area above the standard fractional Brownian motion.

*Keywords:* integration theory; non-commutative fractional Brownian motion; non-commutative stochastic calculus

## References

[1] Biane, P. and Speicher, R. (1998). Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. *Probab. Theory Related Fields* **112** 373–409. MR1660906
[2] Bożejko, M., Kümmerer, B. and Speicher, R. (1997). $q$-Gaussian processes: Non-commutative and classical aspects. *Comm. Math. Phys.* **185** 129–154. MR1463036
[3] Coutin, L. and Qian, Z. (2002). Stochastic analysis, rough path analysis and fractional Brownian motions. *Probab. Theory Related Fields* **122** 108–140. MR1883719
[4] Deya, A. and Schott, R. (2013). On the rough-paths approach to non-commutative stochastic calculus. *J. Funct. Anal.* **265** 594–628. MR3062538
[5] Deya, A. and Schott, R. (2018). On stochastic calculus with respect to $q$-Brownian motion. *J. Funct. Anal.* **274** 1047–1075. MR3743190
[6] Deya, A. and Schott, R. (2019). Supplement to "Integration with respect to the non-commutative fractional Brownian motion." DOI:10.3150/18-BEJ1048SUPP.
[7] Donati-Martin, C. (2003). Stochastic integration with respect to $q$ Brownian motion. *Probab. Theory Related Fields* **125** 77–95. MR1952458
[8] Friz, P. and Victoir, N. (2010). Differential equations driven by Gaussian signals. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** 369–413. MR2667703
[9] Friz, P.K. and Hairer, M. (2014). *A Course on Rough Paths: With an Introduction to Regularity Structures. Universitext*. Cham: Springer. MR3289027
[10] Friz, P.K. and Victoir, N.B. (2010). *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications. Cambridge Studies in Advanced Mathematics* **120**. Cambridge: Cambridge Univ. Press. MR2604669

1350-7265 

[11] Gubinelli, M. (2004). Controlling rough paths. *J. Funct. Anal.* **216** 86–140. MR2091358
[12] Lyons, T. and Qian, Z. (2002). *System Control and Rough Paths. Oxford Mathematical Monographs. Oxford Science Publications*. Oxford: Oxford Univ. Press. MR2036784
[13] Nica, A. and Speicher, R. (2006). *Lectures on the Combinatorics of Free Probability. London Mathematical Society Lecture Note Series* **335**. Cambridge: Cambridge Univ. Press. MR2266879
[14] Nourdin, I. (2012). *Selected Aspects of Fractional Brownian Motion. Bocconi & Springer Series* **4**. New York: Springer. MR3076266
[15] Nourdin, I. and Taqqu, M.S. (2014). Central and non-central limit theorems in a free probability setting. *J. Theoret. Probab.* **27** 220–248. MR3174224
[16] Nualart, D. and Tindel, S. (2011). A construction of the rough path above fractional Brownian motion using Volterra's representation. *Ann. Probab.* **39** 1061–1096. MR2789583
[17] Pardo, J.C., Pérez, J. and Pérez-Abreu, V. (2016). A random matrix approximation for the non-commutative fractional Brownian motion. *J. Theoret. Probab.* **29** 1581–1598. MR3571255
[18] Voiculescu, D. (1991). Limit laws for random matrices and free products. *Invent. Math.* **104** 201–220. MR1094052
[19] Young, L.C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** 251–282. MR1555421

*Bernoulli* **25**(3), 2019, 2163–2182
https://doi.org/10.3150/18-BEJ1049

# Construction of marginally coupled designs by subspace theory

YUANZHEN HE[1], C. DEVON LIN[2] and FASHENG SUN[3,4]

[1]*Beijing Normal University, Beijing, China*
[2]*Queen's University, Kingston, Canada*
[3]*KLAS, Changchun, China*
[4]*Northeast Normal University, Changchun, China. E-mail: sunfs359@nenu.edu.cn*

Recent researches on designs for computer experiments with both qualitative and quantitative factors have advocated the use of marginally coupled designs. This paper proposes a general method of constructing such designs for which the designs for qualitative factors are multi-level orthogonal arrays and the designs for quantitative factors are Latin hypercubes with desirable space-filling properties. Two cases are introduced for which we can obtain the guaranteed low-dimensional space-filling property for quantitative factors. Theoretical results on the proposed constructions are derived. For practical use, some constructed designs for three-level qualitative factors are tabulated.

*Keywords:* cascading Latin hypercube; computer experiment; Latin hypercube; lower-dimensional projection; orthogonal array

## References

[1] Deng, X., Hung, Y. and Lin, C.D. (2015). Design for computer experiments with qualitative and quantitative factors. *Statist*. *Sinica* **25** 1567–1581. MR3409081
[2] Deng, X., Lin, C.D., Liu, K.-W. and Rowe, R.K. (2017). Additive Gaussian process for computer models with qualitative and quantitative factors. *Technometrics* **59** 283–292. MR3677960
[3] Draguljić, D., Santner, T.J. and Dean, A.M. (2012). Noncollapsing space-filling designs for bounded nonrectangular regions. *Technometrics* **54** 169–178. MR2929437
[4] Han, G., Santner, T.J., Notz, W.I. and Bartel, D.L. (2009). Prediction for computer experiments having quantitative and qualitative input variables. *Technometrics* **51** 278–288. MR2751072
[5] Handcock, M.S. (1991). On cascading Latin hypercube designs and additive models for experiments. *Comm*. *Statist*. *Theory Methods* **20** 417–439. MR1130942
[6] He, Y., Lin, C.D. and Sun, F. (2017). On construction of marginally coupled designs. *Statist*. *Sinica* **27** 665–683. MR3674691
[7] He, Y., Lin, C.D., Sun, F. and Lv, B. (2017). Marginally coupled designs for two-level qualitative factors. *J*. *Statist*. *Plann*. *Inference* **187** 103–108. MR3638045
[8] Hedayat, A.S., Sloane, N.J.A. and Stufken, J. (1999). *Orthogonal Arrays*: *Theory and Applications*. *Springer Series in Statistics*. New York: Springer. With a foreword by C.R. Rao. MR1693498
[9] Horn, R.A. and Johnson, C.R. (2013). *Matrix Analysis*, 2nd ed. Cambridge: Cambridge Univ. Press. MR2978290
[10] Huang, H., Lin, D.K.J., Liu, M.-Q. and Yang, J.-F. (2016). Computer experiments with both qualitative and quantitative variables. *Technometrics* **58** 495–507. MR3556617

[11] Joseph, V.R., Gul, E. and Ba, S. (2015). Maximum projection designs for computer experiments. *Biometrika* **102** 371–380. MR3371010

[12] Leary, S., Bhaskar, A. and Keane, A. (2003). Optimal orthogonal-array-based Latin hypercubes. *J. Appl. Stat.* **30** 585–598. MR1969503

[13] Lin, C.D. and Tang, B. (2015). Latin hypercubes and space-filling designs. In *Handbook of Design and Analysis of Experiments* (D. Bingham, A. Dean, M. Morris and J. Stufken, eds.). *Chapman & Hall/CRC Handb. Mod. Stat. Methods* 593–625. Boca Raton, FL: CRC Press. MR3699363

[14] McKay, M.D., Beckman, R.J. and Conover, W.J. (1979). A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics* **21** 239–245. MR0533252

[15] Qian, P.Z.G. and Wu, C.F.J. (2009). Sliced space-filling designs. *Biometrika* **96** 945–956. MR2767280

[16] Qian, P.Z.G., Wu, H. and Wu, C.F.J. (2008). Gaussian process models for computer experiments with qualitative and quantitative factors. *Technometrics* **50** 383–396. MR2457574

[17] Rawlinson, J.J., Furman, B.D., Li, S., Wright, T.M. and Bartel, D.L. (2006). Retrieval, experimental, and computational assessment of the performance of total knee replacements. *J. Orthop. Res.* **24** 1384–1394.

[18] Sun, F. and Tang, B. (2017). A method of constructing space-filling orthogonal designs. *J. Amer. Statist. Assoc.* **112** 683–689. MR3671762

[19] Tang, B. (1993). Orthogonal array-based Latin hypercubes. *J. Amer. Statist. Assoc.* **88** 1392–1397. MR1245375

[20] Wu, C.F.J. and Hamada, M.S. (2009). *Experiments: Planning, Analysis, and Optimization*, 2nd ed. *Wiley Series in Probability and Statistics*. Hoboken, NJ: Wiley. MR2583259

[21] Xie, H., Xiong, S., Qian, P.Z.G. and Wu, C.F.J. (2014). General sliced Latin hypercube designs. *Statist. Sinica* **24** 1239–1256. MR3241286

[22] Zhou, Q., Jin, T., Qian, P.Z.G. and Zhou, S. (2016). Bi-directional sliced Latin hypercube designs. *Statist. Sinica* **26** 653–674. MR3497765

[23] Zhou, Q., Qian, P.Z.G. and Zhou, S. (2011). A simple approach to emulation for computer models with qualitative and quantitative factors. *Technometrics* **53** 266–273. MR2857704

*Bernoulli* **25**(3), 2019, 2183–2205
https://doi.org/10.3150/18-BEJ1050

# Consistency of Bayesian nonparametric inference for discretely observed jump diffusions

JERE KOSKELA[1,*] DARIO SPANÒ[1,**] and PAUL A. JENKINS[2]

[1]*Department of Statistics, University of Warwick, Coventry CV4 7AL, UK.*
*E-mail:* [*]*j.koskela@warwick.ac.uk;* [**]*d.spano@warwick.ac.uk*
[2]*Departments of Statistics and Computer Science, University of Warwick, Coventry CV4 7AL, UK.*
*E-mail:* *p.jenkins@warwick.ac.uk*

We introduce verifiable criteria for weak posterior consistency of Bayesian nonparametric inference for jump diffusions with unit diffusion coefficient and uniformly Lipschitz drift and jump coefficients in arbitrary dimension. The criteria are expressed in terms of coefficients of the SDEs describing the process, and do not depend on intractable quantities such as transition densities. We also show that priors built from discrete nets, wavelet expansions, and Dirichlet mixture models satisfy our conditions. This generalises known results by incorporating jumps into previous work on unit diffusions with uniformly Lipschitz drift coefficients.

*Keywords:* Bayesian statistics; Dirichlet mixture model prior; discrete net prior; jump diffusion; nonparametric inference; posterior consistency

## References

[1] Aase, K.K. and Guttorp, P. (1987). Estimation in models for security prices. *Scand. Actuar. J.* **3–4** 211–224. MR0943582
[2] Aït-Sahalia, Y. (2008). Closed-form likelihood expansions for multivariate diffusions. *Ann. Statist.* **36** 906–937. MR2396819
[3] Applebaum, D. (2004). *Lévy Processes and Stochastic Calculus. Cambridge Studies in Advanced Mathematics* **93**. Cambridge: Cambridge Univ. Press. MR2072890
[4] Au, S.P., Haddad, A.H. and Poor, H.V. (1982). A state estimation algorithm for linear systems driven simultaneously by Wiener and Poisson processes. *IEEE Trans. Automat. Control* **27** 617–626. MR0680320
[5] Bardhan, I. and Chao, X. (1993). Pricing options on securities with discontinuous returns. *Stochastic Process. Appl.* **48** 123–137. MR1237171
[6] Bertoin, J. and Le Gall, J.-F. (2003). Stochastic flows associated to coalescent processes. *Probab. Theory Related Fields* **126** 261–288. MR1990057
[7] Bhattacharya, A. and Dunson, D.B. (2012). Strong consistency of nonparametric Bayes density estimation on compact metric spaces with applications to specific manifolds. *Ann. Inst. Statist. Math.* **64** 687–714. MR2927766
[8] Birkner, M., Blath, J., Möhle, M., Steinrücken, M. and Tams, J. (2009). A modified lookdown construction for the Xi–Fleming–Viot process with mutation and populations with recurrent bottlenecks. *ALEA Lat. Am. J. Probab. Math. Stat.* **6** 25–61. MR2485878

1350-7265 

[9] Bodo, B.A., Thompson, M.E. and Unny, T.E. (1987). A review of stochastic differential equations for applications in hydrology. *Stoch*. *Hydrol*. *Hydraul*. **2** 81–100.
[10] Casella, B. and Roberts, G.O. (2011). Exact simulation of jump-diffusion processes with Monte Carlo applications. *Methodol*. *Comput*. *Appl*. *Probab*. **13** 449–473. MR2822390
[11] Chen, L. and Filipović, D. (2005). A simple model for credit migration and spread curves. *Finance Stoch*. **9** 211–231. MR2211125
[12] Cheridito, P., Filipović, D. and Yor, M. (2005). Equivalent and absolutely continuous measure changes for jump-diffusion processes. *Ann*. *Appl*. *Probab*. **15** 1713–1732. MR2152242
[13] Comte, F., Genon-Catalot, V. and Rozenholc, Y. (2007). Penalized nonparametric mean square estimation of the coefficients of diffusion processes. *Bernoulli* **13** 514–543. MR2331262
[14] Dalalyan, A. and Reiß, M. (2007). Asymptotic statistical equivalence for ergodic diffusions: The multidimensional case. *Probab*. *Theory Related Fields* **137** 25–47. MR2278451
[15] Diaconis, P. and Freedman, D. (1986). On the consistency of Bayes estimates. *Ann*. *Statist*. **14** 1–67. MR0829555
[16] Dudley, R.M. (2002). *Real Analysis and Probability*. *Cambridge Studies in Advanced Mathematics* **74**. Cambridge: Cambridge Univ. Press. Revised reprint of the 1989 original. MR1932358
[17] Ferguson, T.S. (1973). A Bayesian analysis of some nonparametric problems. *Ann*. *Statist*. **1** 209–230. MR0350949
[18] Filipović, D., Cheridito, P. and Kimmel, R.L. (2007). Market price of risk specifications for affine models: Theory and evidence. *J*. *Financ*. *Econ*. **83** 123–170.
[19] Fornaro, S. (2004). Regularity properties for second order partial differential operators with unbounded coefficients. Ph.D. thesis, Univ. del Salento.
[20] Ghosal, S., Ghosh, J.K. and Ramamoorthi, R.V. (1997). Non-informative priors via sieves and packing numbers. In *Advances in Statistical Decision Theory and Applications*. *Stat*. *Ind*. *Technol*. 119–132. Boston, MA: Birkhäuser. MR1479180
[21] Ghosal, S. and Tang, Y. (2006). Bayesian consistency for Markov processes. *Sankhyā* **68** 227–239. MR2303082
[22] Ghosal, S. and van der Vaart, A. (2007). Convergence rates of posterior distributions for non-i.i.d. observations. *Ann*. *Statist*. **35** 192–223. MR2332274
[23] Gobet, E., Hoffmann, M. and Reiß, M. (2004). Nonparametric estimation of scalar diffusions based on low frequency data. *Ann*. *Statist*. **32** 2223–2253. MR2102509
[24] Gonçalves, F.B. (2011). Exact simulation and Monte Carlo inference for jump-diffusion processes. Ph.D. thesis, Univ. Warwick.
[25] Gonçalves, F.B. and Roberts, G.O. (2014). Exact simulation problems for jump-diffusions. *Methodol*. *Comput*. *Appl*. *Probab*. **16** 907–930. MR3270601
[26] Gugushvili, S. and Spreij, P. (2014). Nonparametric Bayesian drift estimation for multidimensional stochastic differential equations. *Lith*. *Math*. *J*. **54** 127–141. MR3212631
[27] Jacod, J. (2000). Non-parametric kernel estimation of the coefficient of a diffusion. *Scand*. *J*. *Stat*. **27** 83–96. MR1774045
[28] Kallianpur, G. (1992). Stochastic differential equation models for spatially distributed neurons and propagation of chaos for interacting systems. *Math*. *Biosci*. **112** 207–224. MR1196373
[29] Kallianpur, G. and Xiong, J. (1994). Asymptotic behavior of a system of interacting nuclear-space-valued stochastic differential equations driven by Poisson random measures. *Appl*. *Math*. *Optim*. **30** 175–201. MR1284327
[30] Kolokoltsov, V.N. (2004). On Markov processes with decomposable pseudo-differential generators. *Stoch*. *Stoch*. *Rep*. **76** 1–44. MR2038027
[31] Lijoi, A., Prünster, I. and Walker, S.G. (2004). Extending Doob's consistency theorem to nonparametric densities. *Bernoulli* **10** 651–663. MR2076067

[32] Lo, A.Y. (1984). On a class of Bayesian nonparametric estimates. I. Density estimates. *Ann. Statist.* **12** 351–357. MR0733519
[33] Masuda, H. (2007). Ergodicity and exponential $\beta$-mixing bounds for multidimensional diffusions with jumps. *Stochastic Process. Appl.* **117** 35–56. MR2287102
[34] Masuda, H. (2009). Erratum to: "Ergodicity and exponential $\beta$-mixing bound for multidimensional diffusions with jumps" [Stochastic Process. Appl. 117 (2007) 35–56] [MR2287102]. *Stochastic Process. Appl.* **119** 676–678. MR2494009
[35] Merton, R.C. (1976). Option pricing when underlying stock returns are discontinuous. *J. Financ. Econ.* **3** 125–144.
[36] Meyer, Y. (1992). *Wavelets and Operators. Cambridge Studies in Advanced Mathematics* **37**. Cambridge: Cambridge Univ. Press. MR1228209
[37] Nickl, R. and Söhl, J. (2017). Nonparametric Bayesian posterior contraction rates for discretely observed scalar diffusions. *Ann. Statist.* **45** 1664–1693. MR3670192
[38] Panzar, L. and van Zanten, H. (2009). Nonparametric Bayesian inference for ergodic diffusions. *J. Statist. Plann. Inference* **139** 4193–4199. MR2558361
[39] Papaspiliopoulos, O., Pokern, Y., Roberts, G.O. and Stuart, A.M. (2012). Nonparametric estimation of diffusions: A differential equations approach. *Biometrika* **99** 511–531. MR2966767
[40] Pokern, Y., Stuart, A.M. and van Zanten, J.H. (2013). Posterior consistency via precision operators for Bayesian nonparametric drift estimation in SDEs. *Stochastic Process. Appl.* **123** 603–628. MR3003365
[41] Pollock, M. (2015). On the exact simulation of (jump) diffusion bridges. In *Proceedings of the* 2015 *Winter Simulation Conference* IEEE Press.
[42] Pollock, M., Johansen, A.M. and Roberts, G.O. (2016). On the exact and $\varepsilon$-strong simulation of (jump) diffusions. *Bernoulli* **22** 794–856. MR3449801
[43] Protter, P.E. (2005). *Stochastic Integration and Differential Equations*, 2nd ed. *Stochastic Modelling and Applied Probability* **21**. Berlin: Springer. Version 2.1, Corrected third printing. MR2273672
[44] Ruggeri, F. and Vidakovic, B. (2005). Bayesian modeling in the wavelet domain. In *Bayesian Thinking: Modeling and Computation. Handbook of Statist.* **25** 315–338. Amsterdam: Elsevier/North-Holland. MR2490531
[45] Schilling, R.L. and Wang, J. (2013). Some theorems on Feller processes: Transience, local times and ultracontractivity. *Trans. Amer. Math. Soc.* **365** 3255–3286. MR3034465
[46] Schmisser, E. (2013). Penalized nonparametric drift estimation for a multidimensional diffusion process. *Statistics* **47** 61–84. MR3023015
[47] Stramer, O. and Tweedie, R.L. (1997). Existence and stability of weak solutions to stochastic differential equations with non-smooth coefficients. *Statist. Sinica* **7** 577–593. MR1467449
[48] Tang, Y. and Ghosal, S. (2007). Posterior consistency of Dirichlet mixtures for estimating a transition density. *J. Statist. Plann. Inference* **137** 1711–1726. MR2323858
[49] van der Meulen, F., Schauer, M. and van Zanten, H. (2014). Reversible jump MCMC for nonparametric drift estimation for diffusion processes. *Comput. Statist. Data Anal.* **71** 615–632. MR3131993
[50] van der Meulen, F. and van Zanten, H. (2013). Consistent nonparametric Bayesian inference for discretely observed scalar diffusions. *Bernoulli* **19** 44–63. MR3019485
[51] van der Meulen, F.H., van der Vaart, A.W. and van Zanten, J.H. (2006). Convergence rates of posterior distributions for Brownian semimartingale models. *Bernoulli* **12** 863–888. MR2265666
[52] van Zanten, H. (2013). Nonparametric Bayesian methods for one-dimensional diffusion models. *Math. Biosci.* **243** 215–222. MR3065207
[53] Walker, S. (2004). New approaches to Bayesian consistency. *Ann. Statist.* **32** 2028–2043. MR2102501
[54] Wang, J. (2010). Regularity of semigroups generated by Lévy type operators via coupling. *Stochastic Process. Appl.* **120** 1680–1700. MR2673970

[55] Yu, J. (2007). Closed-form likelihood approximation and estimation of jump-diffusions with an application to the realignment risk of the Chinese Yuan. *J. Econometrics* **141** 1245–1280. MR2413501

*Bernoulli* **25**(3), 2019, 2206–2244
https://doi.org/10.3150/18-BEJ1051

# On the risk of convex-constrained least squares estimators under misspecification

BILLY FANG[*] and ADITYANAND GUNTUBOYINA[**]

*Department of Statistics, University of California, Berkeley CA 94720, USA.*
*E-mail:* [*]*blfang@berkeley.edu;* [**]*aditya@stat.berkeley.edu*

We consider the problem of estimating the mean of a noisy vector. When the mean lies in a convex constraint set, the least squares projection of the random vector onto the set is a natural estimator. Properties of the risk of this estimator, such as its asymptotic behavior as the noise tends to zero, have been well studied. We instead study the behavior of this estimator under misspecification, that is, without the assumption that the mean lies in the constraint set. For appropriately defined notions of risk in the misspecified setting, we prove a generalization of a low noise characterization of the risk due to [*Found. Comput. Math.* **16** (2016) 965–1029] in the case of a polyhedral constraint set. An interesting consequence of our results is that the risk can be much smaller in the misspecified setting than in the well-specified setting. We also discuss consequences of our result for isotonic regression.

*Keywords:* convex constraint; isotonic regression; least squares; misspecification; statistical dimension

## References

[1] Amelunxen, D. and Lotz, M. (2017). Intrinsic volumes of polyhedral cones: A combinatorial perspective. *Discrete Comput. Geom.* **58** 371–409. MR3679941

[2] Amelunxen, D., Lotz, M., McCoy, M.B. and Tropp, J.A. (2014). Living on the edge: Phase transitions in convex programs with random data. *Inf. Inference* **3** 224–294. MR3311453

[3] Bellec, P.C. (2018). Sharp oracle inequalities for least squares estimators in shape restricted regression. *Ann. Statist.* **46** 745–780. MR3782383

[4] Groeneboom, P. and Jongbloed, G. (2014). *Nonparametric Estimation Under Shape Constraints: Estimators, Algorithms and Asymptotics*. *Cambridge Series in Statistical and Probabilistic Mathematics* **38**. New York: Cambridge Univ. Press. MR3445293

[5] Hiriart-Urruty, J.-B. and Lemaréchal, C. (2001). *Fundamentals of Convex Analysis*. *Grundlehren Text Editions*. Berlin: Springer. Abridged version of *Convex analysis and minimization algorithms. I* [Springer, Berlin, 1993; MR1261420 (95m:90001)] and *II* [ibid.; MR1295240 (95m:90002)]. MR1865628

[6] Jankowski, H. (2014). Convergence of linear functionals of the Grenander estimator under misspecification. *Ann. Statist.* **42** 625–653. MR3210981

[7] Klivans, C.J. and Swartz, E. (2011). Projection volumes of hyperplane arrangements. *Discrete Comput. Geom.* **46** 417–426. MR2826960

[8] Oymak, S. and Hassibi, B. (2016). Sharp MSE bounds for proximal denoising. *Found. Comput. Math.* **16** 965–1029. MR3529131

[9] Pal, J.K. (2008). Spiking problem in monotone regression: Penalized residual sum of squares. *Statist. Probab. Lett.* **78** 1548–1556. MR2453808

1350-7265 

[10] Robertson, T., Wright, F.T. and Dykstra, R.L. (1988). *Order Restricted Statistical Inference*. *Wiley Series in Probability and Mathematical Statistics*: *Probability and Mathematical Statistics*. Chichester: Wiley. MR0961262

[11] Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. *J. Roy. Statist. Soc. Ser. B* **58** 267–288. MR1379242

[12] Wu, J., Meyer, M.C. and Opsomer, J.D. (2015). Penalized isotonic regression. *J. Statist. Plann. Inference* **161** 12–24. MR3316548

[13] Zhang, C.-H. (2002). Risk bounds in isotonic regression. *Ann. Statist.* **30** 528–555. MR1902898

*Bernoulli* **25**(3), 2019, 2245–2278
https://doi.org/10.3150/18-BEJ1052

# A central limit theorem for the realised covariation of a bivariate Brownian semistationary process

ANDREA GRANELLI* and ALMUT E.D. VERAART**

*Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom. E-mail:* *andrea.granelli@gmail.com; **a.veraart@imperial.ac.uk

This article presents a weak law of large numbers and a central limit theorem for the scaled realised covariation of a bivariate Brownian semistationary process. The novelty of our results lies in the fact that we derive the suitable asymptotic theory both in a multivariate setting and outside the classical semimartingale framework. The proofs rely heavily on recent developments in Malliavin calculus.

*Keywords:* bivariate Brownian semistationary process; central limit theorem; fourth moment theorem; high frequency data; moving average process; multivariate setting; stable convergence

## References

[1] Barndorff-Nielsen, O.E. (2016). Gamma kernels and BSS/LSS processes. In *Advanced Modelling in Mathematical Finance*. *Springer Proc*. *Math*. *Stat*. **189** 41–61. Cham: Springer. MR3630532
[2] Barndorff-Nielsen, O.E., Benth, F.E. and Veraart, A.E.D. (2013). Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. *Bernoulli* **19** 803–845. MR3079297
[3] Barndorff-Nielsen, O.E., Corcuera, J.M. and Podolskij, M. (2009). Power variation for Gaussian processes with stationary increments. *Stochastic Process*. *Appl*. **119** 1845–1865. MR2519347
[4] Barndorff-Nielsen, O.E., Corcuera, J.M. and Podolskij, M. (2011). Multipower variation for Brownian semistationary processes. *Bernoulli* **17** 1159–1194. MR2854768
[5] Barndorff-Nielsen, O.E., Corcuera, J.M. and Podolskij, M. (2013). Limit theorems for functionals of higher order differences of Brownian semi-stationary processes. In *Prokhorov and Contemporary Probability Theory*. *Springer Proc*. *Math*. *Stat*. **33** 69–96. Heidelberg: Springer. MR3070467
[6] Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M. and Woerner, J.H.C. (2009). Bipower variation for Gaussian processes with stationary increments. *J*. *Appl*. *Probab*. **46** 132–150. MR2508510
[7] Barndorff-Nielsen, O.E., Graversen, S.E., Jacod, J., Podolskij, M. and Shephard, N. (2006). A central limit theorem for realised power and bipower variations of continuous semimartingales. In *From Stochastic Calculus to Mathematical Finance* 33–68. Berlin: Springer. MR2233534
[8] Barndorff-Nielsen, O.E. and Schmiegel, J. (2009). Brownian semistationary processes and volatility/intermittency. In *Advanced Financial Modelling*. *Radon Ser*. *Comput*. *Appl*. *Math*. **8** 1–25. Berlin: Walter de Gruyter. MR2648456
[9] Barndorff-Nielsen, O.E. and Shephard, N. (2004). Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics. *Econometrica* **72** 885–925. MR2051439
[10] Barndorff-Nielsen, O.E. and Shephard, N. (2004). Power and bipower variation with stochastic volatility and jumps. *J*. *Financ*. *Econom*. **2** 1–37.

1350-7265 

[11] Barndorff-Nielsen, O.E. and Shephard, N. (2006). Econometrics of testing for jumps in financial economics using bipower variation. *J. Financ. Econom.* **4** 1–30.
[12] Barndorff-Nielsen, O.E., Shephard, N. and Winkel, M. (2006). Limit theorems for multipower variation in the presence of jumps. *Stochastic Process. Appl.* **116** 796–806. MR2218336
[13] Basse, A. (2008). Gaussian moving averages and semimartingales. *Electron. J. Probab.* **13** 1140–1165. MR2424990
[14] Bayer, C., Friz, P. and Gatheral, J. (2016). Pricing under rough volatility. *Quant. Finance* **16** 887–904. MR3494612
[15] Bennedsen, M., Lunde, A. and Pakkanen, M.S. (2017). Hybrid scheme for Brownian semistationary processes. *Finance Stoch.* **21** 931–965. MR3723378
[16] Bernstein, S. (1927). Sur l'extension du théoréme limite du calcul des probabilités aux sommes de quantités dépendantes. *Math. Ann.* **97** 1–59. MR1512353
[17] Billingsley, P. (2009). *Convergence of Probability Measures* **493**. John Wiley & Sons. MR0233396
[18] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1989). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge: Cambridge Univ. Press. MR1015093
[19] Corcuera, J.M. (2012). New central limit theorems for functionals of Gaussian processes and their applications. *Methodol. Comput. Appl. Probab.* **14** 477–500. MR2966306
[20] Corcuera, J.M., Hedevang, E., Pakkanen, M.S. and Podolskij, M. (2013). Asymptotic theory for Brownian semi-stationary processes with application to turbulence. *Stochastic Process. Appl.* **123** 2552–2574. MR3054536
[21] Corcuera, J.M., Nualart, D. and Podolskij, M. (2014). Asymptotics of weighted random sums. *Commun. Appl. Ind. Math.* **6** e–486, 11. MR3277312
[22] Corcuera, J.M., Nualart, D. and Woerner, J.H.C. (2006). Power variation of some integral fractional processes. *Bernoulli* **12** 713–735. MR2248234
[23] Föllmer, H., Protter, P. and Shiryayev, A.N. (1995). Quadratic covariation and an extension of Itô's formula. *Bernoulli* **1** 149–169. MR1354459
[24] Gatheral, J., Jaisson, T. and Rosenbaum, M. (2018). Volatility is rough. *Quant. Finance* **18** 933–949. MR3805308
[25] Granelli, A. and Veraart, A.E.D. (2017). A weak law of large numbers for estimating the correlation in bivariate Brownian semistationary processes. ArXiv.
[26] Granelli, A. and Veraart, A.E.D. (2019). Supplement to "A central limit theorem for the realised covariation of a bivariate Brownian semistationary process." DOI:10.3150/18-BEJ1052SUPP.
[27] Jacod, J. (2008). Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Process. Appl.* **118** 517–559. MR2394762
[28] Jacod, J. and Protter, P. (2012). *Discretization of Processes. Stochastic Modelling and Applied Probability* **67**. Heidelberg: Springer. MR2859096
[29] Kinnebrock, S. and Podolskij, M. (2008). A note on the central limit theorem for bipower variation of general functions. *Stochastic Process. Appl.* **118** 1056–1070. MR2418258
[30] Knight, F.B. (1992). *Foundations of the Prediction Process. Oxford Studies in Probability* **1**. New York: The Clarendon Press, Oxford Univ. Press. Oxford Science Publications. MR1168699
[31] Lépingle, D. (1976). La variation d'ordre $p$ des semi-martingales. *Z. Wahrsch. Verw. Gebiete* **36** 295–316. MR0420837
[32] Nourdin, I. and Peccati, G. (2012). *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality. Cambridge Tracts in Mathematics* **192**. Cambridge: Cambridge Univ. Press. MR2962301
[33] Nualart, D. and Peccati, G. (2005). Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33** 177–193. MR2118863

[34] Pakkanen, M.S. (2011). Brownian semistationary processes and conditional full support. *Int*. *J*. *Theor*. *Appl*. *Finance* **14** 579–586. MR2819842
[35] Protter, P.E. (2005). *Stochastic Integration and Differential Equations*, 2nd edition. Version 2.1, Corrected third printing. *Stochastic Modelling and Applied Probability* **21**. Berlin: Springer. MR2273672
[36] Vetter, M. (2010). Limit theorems for bipower variation of semimartingales. *Stochastic Process*. *Appl*. **120** 22–38. MR2565850

*Bernoulli* **25**(3), 2019, 2279–2300
https://doi.org/10.3150/18-BEJ1054

# The first order correction to harmonic measure for random walks of rotationally invariant step distribution

LONGMIN WANG[1,*], KAINAN XIANG[2] and LANG ZOU[1,**]

[1]*School of Mathematical Sciences, LPMC, Nankai University. 94 Weijin Road, Tianjin 300071, P.R. China. E-mail:* [*]*wanglm@nankai.edu.cn;* [**]*lang.zou.proba@gmail.com*
[2]*School of Mathematics and Computational Science, Xiangtan University Xiangtan City 210000, Hunan Province, P. R. China. E-mail:* *kainan.xiang@xtu.edu.cn*

Let $D \subset \mathbb{R}^d$ $(d \geq 2)$ be an open simply-connected bounded domain with smooth boundary $\partial D$ and $\mathbf{0} = (0, \dots, 0) \in D$. Fix any rotationally invariant probability $\mu$ on closed unit ball $\{z \in \mathbb{R}^d : |z| \leq 1\}$ with $\mu(\{\mathbf{0}\}) < 1$. Let $\{S_n^\mu\}_{n=0}^\infty$ be the random walk with step-distribution $\mu$ starting at $\mathbf{0}$. Denote by $\omega_\delta(\mathbf{0}, \mathrm{d}z; D)$ the discrete harmonic measure for $\{\delta S_n^\mu\}_{n=0}^\infty$ $(\delta > 0)$ exiting from $D$, which is viewed as a probability on $\partial D$ by projecting suitably the first exiting point to $\partial D$. Denote by $\omega(\mathbf{0}, \mathrm{d}z; D)$ the harmonic measure for the $d$-dimensional standard Brownian motion exiting from $D$. Then in the weak convergence topology,

$$\lim_{\delta \to 0} \frac{1}{\delta} \big[\omega_\delta(\mathbf{0}, \mathrm{d}z; D) - \omega(\mathbf{0}, \mathrm{d}z; D)\big] = c_\mu \rho_D(z) \, |\mathrm{d}z|,$$

where $\rho_D(\cdot)$ is a smooth function depending on $D$ but not on $\mu$, $c_\mu$ is a constant depending only on $\mu$, and $|\mathrm{d}z|$ is the Lebesgue measure with respect to $\partial D$. Additionally, $\rho_D(z)$ is determined by the following equation: For any smooth function $g$ on $\partial D$,

$$\int_{\partial D} g(z) \rho_D(z) \, |\mathrm{d}z| = \int_{\partial D} \frac{\partial f}{\partial \mathbf{n}_z}(z) H_D(\mathbf{0}, z) \, |\mathrm{d}z|,$$

where $f$ is the harmonic function in $D$ with boundary values given by $g$, $H_D(\mathbf{0}, z)$ is the Poisson kernel and derivative $\frac{\partial f}{\partial \mathbf{n}_z}$ is with respect to the inward unit normal $\mathbf{n}_z$ at $z \in \partial D$.

*Keywords:* discrete harmonic measure; first order correction; harmonic measure; random walk

## References

[1] Axler, S., Bourdon, P. and Ramey, W. (2001). *Harmonic Function Theory*, 2nd ed. *Graduate Texts in Mathematics* **137**. New York: Springer. MR1805196
[2] Brychkov, Y.A. (2008). *Handbook of Special Functions*: *Derivatives*, *Integrals*, *Series and Other Formulas*. Boca Raton, FL: CRC Press. MR2460394
[3] Chelkak, D. and Smirnov, S. (2011). Discrete complex analysis on isoradial graphs. *Adv. Math.* **228** 1590–1630. MR2824564
[4] Dai, Y. (2017). The exit distribution for smart kinetic walk with symmetric and asymmetric transition probability. *J. Stat. Phys.* **166** 1455–1463. MR3612235

[5] Garnett, J.B. and Marshall, D.E. (2005). *Harmonic Measure*. *New Mathematical Monographs* **2**. Cambridge: Cambridge Univ. Press. MR2150803

[6] Gasper, G. and Rahman, M. (2004). *Basic Hypergeometric Series*, 2nd ed. *Encyclopedia of Mathematics and Its Applications* **96**. Cambridge: Cambridge Univ. Press. MR2128719

[7] Gilbarg, D. and Trudinger, N.S. (2001). *Elliptic Partial Differential Equations of Second Order*. Berlin: Springer. MR1814364

[8] Hughes, B.D. (1995). *Random Walks and Random Environments*: *Random Walks*. *Oxford Science Publications* **1**. New York: Clarendon Press. MR1341369

[9] Jerison, D. (1990). Regularity of the Poisson kernel and free boundary problems. *Colloq*. *Math*. **60/61** 547–568. MR1096396

[10] Jiang, J. and Kennedy, T. (2017). The difference between a discrete and continuous harmonic measure. *J*. *Theoret*. *Probab*. **30** 1424–1444. MR3736178

[11] Karatzas, I. and Shreve, S.E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. New York: Springer. MR1121940

[12] Kennedy, T. (2016). The first order correction to the exit distribution for some random walks. *J*. *Stat*. *Phys*. **164** 174–189. MR3509053

[13] Lawler, G.F. (1991). *Intersections of Random Walks*. Boston, MA: Birkhäuser. MR1117680

[14] Lawler, G.F. and Limic, V. (2010). *Random Walk*: *A Modern Introduction*. *Cambridge Studies in Advanced Mathematics* **123**. Cambridge: Cambridge Univ. Press. MR2677157

[15] Mörters, P. and Peres, Y. (2010). *Brownian Motion*. *Cambridge Series in Statistical and Probabilistic Mathematics* **30**. Cambridge: Cambridge Univ. Press. MR2604525

[16] Watson, G.N. (1944). *A Treatise on the Theory of Bessel Functions*. Cambridge: Cambridge Univ. Press. MR0010746

[17] Zabczyk, J. (1970). Sur la théorie semi-classique du potentiel pour les processus à accroissements indépendants. *Studia Math*. **35** 227–247. MR0267643

*Bernoulli* **25**(3), 2019, 2301–2329
https://doi.org/10.3150/18-BEJ1055

# Gromov–Hausdorff–Prokhorov convergence of vertex cut-trees of $n$-leaf Galton–Watson trees

HUI HE[1] and MATTHIAS WINKEL[2]

[1]*Laboratory of Mathematics and Complex Systems, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R. China. E-mail: hehui@bnu.edu.cn*
[2]*Department of Statistics, University of Oxford, 24–29 St Giles', Oxford OX1 3LB, UK. E-mail: winkel@stats.ox.ac.uk*

In this paper, we study the vertex cut-trees of Galton–Watson trees conditioned to have $n$ leaves. This notion is a slight variation of Dieuleveut's vertex cut-tree of Galton–Watson trees conditioned to have $n$ vertices. Our main result is a joint Gromov–Hausdorff–Prokhorov convergence in the finite variance case of the Galton–Watson tree and its vertex cut-tree to Bertoin and Miermont's joint distribution of the Brownian CRT and its cut-tree. The methods also apply to the infinite variance case, but the problem to strengthen Dieuleveut's and Bertoin and Miermont's Gromov–Prokhorov convergence to Gromov–Hausdorff–Prokhorov remains open for their models conditioned to have $n$ vertices.

*Keywords:* Continuum Random Tree; cut-tree; fragmentation at nodes; Galton–Watson tree; Gromov–Hausdorff–Prokhorov topology; Invariance Principle; $\mathbb{R}$-tree; stable tree

## References

[1] Abraham, R. and Delmas, J.-F. (2012). Record process on the continuum random tree. *Ann. Probab.* **40** 1167–1211.
[2] Abraham, R. and Delmas, J.-F. (2014). Local limits of conditioned Galton–Watson trees: The condensation case. *Electron. J. Probab.* **19** no. 56, 29. MR3227065
[3] Abraham, R., Delmas, J.-F. and He, H. (2012). Pruning Galton–Watson trees and tree-valued Markov processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 688–705. MR2976559
[4] Abraham, R., Delmas, J.-F. and Hoscheit, P. (2013). A note on the Gromov–Hausdorff–Prokhorov distance between (locally) compact metric measure spaces. *Electron. J. Probab.* **18** no. 14, 21. MR3035742
[5] Addario-Berry, L., Broutin, N. and Holmgren, C. (2014). Cutting down trees with a Markov chainsaw. *Ann. Appl. Probab.* **24** 2297–2339. MR3262504
[6] Aldous, D. (1991). The continuum random tree. I. *Ann. Probab.* **19** 1–28. MR1085326
[7] Aldous, D. (1991). The continuum random tree. II. An overview. In *Stochastic Analysis* (*Durham*, 1990). *London Mathematical Society Lecture Note Series* **167** 23–70. Cambridge Univ. Press, Cambridge. MR1166406
[8] Aldous, D. (1993). The continuum random tree. III. *Ann. Probab.* **21** 248–289. MR1207226
[9] Aldous, D. and Pitman, J. (1998). Tree-valued Markov chains derived from Galton–Watson processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **34** 637–686. MR1641670

[10] Aldous, D. and Pitman, J. (2000). Inhomogeneous continuum random trees and the entrance boundary of the additive coalescent. *Probab. Theory Related Fields* **118** 455–482. MR1808372

[11] Bertoin, J. (2002). Self-similar fragmentations. *Ann. Inst. Henri Poincaré Probab. Stat.* **38** 319–340. MR1899456

[12] Bertoin, J. (2012). Fires on trees. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** 909–921. MR3052398

[13] Bertoin, J. and Miermont, G. (2013). The cut-tree of large Galton–Watson trees and the Brownian CRT. *Ann. Appl. Probab.* **23** 1469–1493. MR3098439

[14] Broutin, N. and Wang, M. (2017). Cutting down **p**-trees and inhomogeneous continuum random trees. *Bernoulli* **23** 2380–2433. MR3648034

[15] Camarri, M. and Pitman, J. (2000). Limit distributions and random trees derived from the birthday problem with unequal probabilities. *Electron. J. Probab.* **5** no. 2, 18. MR1741774

[16] de Raphélis, L. (2017). Scaling limit of multitype Galton–Watson trees with infinitely many types. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** 200–225. MR3606739

[17] Dieuleveut, D. (2015). The vertex-cut-tree of Galton–Watson trees converging to a stable tree. *Ann. Appl. Probab.* **25** 2215–2262. MR3349006

[18] Duquesne, T. (2003). A limit theorem for the contour process of conditioned Galton–Watson trees. *Ann. Probab.* **31** 996–1027. MR1964956

[19] Duquesne, T. and Le Gall, J.-F. (2002). Random trees, Lévy processes and spatial branching processes. *Astérisque* **281** vi+147. MR1954248

[20] Duquesne, T. and Le Gall, J.-F. (2005). Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields* **131** 553–603. MR2147221

[21] Duquesne, T. and Winkel, M. (2012). Hereditary tree growth and Lévy forests. *Stoch. Proc. Appl.* To appear. Available at arXiv:1211.2179.

[22] Duquesne, T. and Winkel, M. (2017). Hereditary tree growth and decompositions. In preparation.

[23] Evans, S.N., Pitman, J. and Winter, A. (2006). Rayleigh processes, real trees, and root growth with re-grafting. *Probab. Theory Related Fields* **134** 81–126. MR2221786

[24] Evans, S.N. and Winter, A. (2006). Subtree prune and regraft: A reversible real tree-valued Markov process. *Ann. Probab.* **34** 918–961. MR2243874

[25] Feller, W. (1971). *An Introduction to Probability Theory and Its Applications. Vol. II*. 2nd ed. New York–London–Sydney: Wiley. MR0270403

[26] Greven, A., Pfaffelhuber, P. and Winter, A. (2009). Convergence in distribution of random metric measure spaces ($\Lambda$-coalescent measure trees). *Probab. Theory Related Fields* **145** 285–322. MR2520129

[27] Haas, B. and Miermont, G. (2004). The genealogy of self-similar fragmentations with negative index as a continuum random tree. *Electron. J. Probab.* **9** 57–97. MR2041829

[28] Haas, B. and Miermont, G. (2012). Scaling limits of Markov branching trees with applications to Galton–Watson and random unordered trees. *Ann. Probab.* **40** 2589–2666. MR3050512

[29] He, H. and Winkel, M. (2014). Invariance principles for pruning processes of Galton–Watson trees. Available at arXiv:1409.1014.

[30] Ibragimov, I.A. and Linnik, Yu.V. (1971). *Independent and Stationary Sequences of Random Variables*. Groningen: Wolters-Noordhoff Publishing. MR0322926

[31] Janson, S. (2006). Random cutting and records in deterministic and random trees. *Random Structures Algorithms* **29** 139–179. MR2245498

[32] Kallenberg, O. (2002). *Foundations of Modern Probability*, 2nd ed. *Probability and Its Applications* (*New York*). New York: Springer. MR1876169

[33] Kesten, H. (1986). Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. Henri Poincaré Probab. Stat.* **22** 425–487. MR0871905

[34] Kortchemski, I. (2012). Invariance principles for Galton–Watson trees conditioned on the number of leaves. *Stochastic Process. Appl.* **122** 3126–3172. MR2946438

[35] Kortchemski, I. (2014). Random stable laminations of the disk. *Ann. Probab.* **42** 725–759. MR3178472

[36] Meir, A. and Moon, J.W. (1970). Cutting down random trees. *J. Aust. Math. Soc.* **11** 313–324. MR0284370

[37] Miermont, G. (2008). Invariance principles for spatial multitype Galton–Watson trees. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** 1128–1161. MR2469338

[38] Miermont, G. (2009). Tessellations of random maps of arbitrary genus. *Ann. Sci. Éc. Norm. Supér.* (4) **42** 725–781. MR2571957

[39] Neveu, J. (1986). Erasing a branching tree. *Adv. in Appl. Probab.* **18** 101–108. MR0868511

[40] Neveu, J. and Pitman, J. (1989). Renewal property of the extrema and tree property of the excursion of a one-dimensional Brownian motion. In *Séminaire de Probabilités, XXIII. Lecture Notes in Math.* **1372** 239–247. Springer, Berlin. MR1022914

[41] Neveu, J. and Pitman, J.W. (1989). The branching process in a Brownian excursion. In *Séminaire de Probabilités, XXIII. Lecture Notes in Math.* **1372** 248–257. Springer, Berlin. MR1022915

[42] Panholzer, A. (2006). Cutting down very simple trees. *Quaest. Math.* **29** 211–227. MR2233368

[43] Pitman, J. (1999). Coalescent random forests. *J. Combin. Theory Ser. A* **85** 165–193. MR1673928

[44] Pitman, J. (2006). *Combinatorial Stochastic Processes. Lecture Notes in Math.* **1875**. Berlin: Springer. MR2245368

[45] Pitman, J. and Rizzolo, D. (2015). Schröder's problems and scaling limits of random trees. *Trans. Amer. Math. Soc.* **367** 6943–6969. MR3378819

[46] Revuz, D. and Yor, M. (1999). *Continuous Martingales and Brownian Motion*, 3rd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **293**. Berlin: Springer. MR1725357

[47] Rizzolo, D. (2015). Scaling limits of Markov branching trees and Galton–Watson trees conditioned on the number of vertices with out-degree in a given set. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 512–532. MR3335013

*Bernoulli* **25**(3), 2019, 2330–2358
https://doi.org/10.3150/18-BEJ1056

# Bayesian mode and maximum estimation and accelerated rates of contraction

WILLIAM WEIMIN YOO[1] and SUBHASHIS GHOSAL[2]

[1]*Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands. E-mail: yooweimin0203@gmail.com*
[2]*Department of Statistics, North Carolina State University, 4276 SAS Hall, 2311 Stinson Drive, Raleigh, NC 27695-8203, USA. E-mail: sghosal@ncsu.edu*

We study the problem of estimating the mode and maximum of an unknown regression function in the presence of noise. We adopt the Bayesian approach by using tensor-product B-splines and endowing the coefficients with Gaussian priors. In the usual fixed-in-advanced sampling plan, we establish posterior contraction rates for mode and maximum and show that they coincide with the minimax rates for this problem. To quantify estimation uncertainty, we construct credible sets for these two quantities that have high coverage probabilities with optimal sizes. If one is allowed to collect data sequentially, we further propose a Bayesian two-stage estimation procedure, where a second stage posterior is built based on samples collected within a credible set constructed from a first stage posterior. Under appropriate conditions on the radius of this credible set, we can accelerate optimal contraction rates from the fixed-in-advanced setting to the minimax sequential rates. A simulation experiment shows that our Bayesian two-stage procedure outperforms single-stage procedure and also slightly improves upon a non-Bayesian two-stage procedure.

*Keywords:* anisotropic Hölder space; credible set; maximum value; mode; nonparametric regression; posterior contraction; sequential; tensor-product B-splines; two-stage

## References

[1] Belitser, E., Ghosal, S. and van Zanten, H. (2012). Optimal two-stage procedures for estimating location and size of the maximum of a multivariate regression function. *Ann. Statist.* **40** 2850–2876. MR3097962
[2] Blum, J.R. (1954). Multidimensional stochastic approximation methods. *Ann. Math. Statist.* **25** 737–744. MR0065092
[3] Castillo, I. and Nickl, R. (2013). Nonparametric Bernstein–von Mises theorems in Gaussian white noise. *Ann. Statist.* **41** 1999–2028. MR3127856
[4] Castillo, I. and Nickl, R. (2014). On the Bernstein–von Mises phenomenon for nonparametric Bayes procedures. *Ann. Statist.* **42** 1941–1969. MR3262473
[5] Chen, H. (1988). Lower rate of convergence for locating a maximum of a function. *Ann. Statist.* **16** 1330–1334. MR0959206
[6] Cox, D.D. (1993). An analysis of Bayesian inference for nonparametric regression. *Ann. Statist.* **21** 903–923. MR1232525
[7] de Boor, C. (2001). *A Practical Guide to Splines*, Revised ed. New York: Springer. MR1900298
[8] Dippon, J. (2003). Accelerated randomized stochastic optimization. *Ann. Statist.* **31** 1260–1281. MR2001650

1350-7265 

[9] Fabian, V. (1967). Stochastic approximation of minima with improved asymptotic speed. *Ann. Math. Statist.* **38** 191–200. MR0207136
[10] Facer, M.R. and Müller, H.-G. (2003). Nonparametric estimation of the location of a maximum in a response surface. *J. Multivariate Anal.* **87** 191–217. MR2007268
[11] Freedman, D. (1999). On the Bernstein–von Mises theorem with infinite-dimensional parameters. *Ann. Statist.* **27** 1119–1140. MR1740119
[12] Ghosal, S. and van der Vaart, A.W. (2017). *Fundamentals of Nonparametric Bayesian Inference*. Cambridge: Cambridge Univ. Press. MR3587782
[13] Hasminskiĭ, R.Z. (1979). Lower bound for the risks of nonparametric estimates of the mode. In *Contributions to Statistics* (J. Jureckova, ed.) 91–97. Dordrecht: Reidel. MR0561262
[14] Jørgensen, M., Nielsen, C.T., Keiding, N. and Skakkeback, N.E. (1985). Parametrische und nichtparametrische Modelle für Wachstumsdaten. In *Neuere Verfahren der nichtparametrischen Statistik* (G.C. Pflug, ed.). *Med. Informatik und Statistik* **60** 74–87. Berlin: Springer.
[15] Kiefer, J. and Wolfowitz, J. (1952). Stochastic estimation of the maximum of a regression function. *Ann. Math. Statist.* **23** 462–466. MR0050243
[16] Kim, J. and Pollard, D. (1990). Cube root asymptotics. *Ann. Statist.* **18** 191–219. MR1041391
[17] Knapik, B.T., van der Vaart, A.W. and van Zanten, J.H. (2011). Bayesian inverse problems with Gaussian priors. *Ann. Statist.* **39** 2626–2657. MR2906881
[18] Lan, Y., Banerjee, M. and Michailidis, G. (2009). Change-point estimation under adaptive sampling. *Ann. Statist.* **37** 1752–1791. MR2533471
[19] Mokkadem, A. and Pelletier, M. (2007). A companion for the Kiefer–Wolfowitz–Blum stochastic approximation algorithm. *Ann. Statist.* **35** 1749–1772. MR2351104
[20] Müller, H.-G. (1985). Kernel estimators of zeros and of location and size of extrema of regression functions. *Scand. J. Stat.* **12** 221–232. MR0817940
[21] Müller, H.-G. (1989). Adaptive nonparametric peak estimation. *Ann. Statist.* **17** 1053–1069. MR1015137
[22] Polyak, B.T. and Tsybakov, A.B. (1990). Optimal order of accuracy of search algorithms in stochastic optimization. *Probl. Inf. Transm.* **26** 126–133. MR1074128
[23] Schumaker, L.L. (2007). *Spline Functions: Basic Theory*, 3rd ed. New York: Cambridge Univ. Press. MR2348176
[24] Shen, W. and Ghosal, S. (2015). Adaptive Bayesian procedures using random series priors. *Scand. J. Statist.* **42** 1194–1213. MR3426318
[25] Shoung, J.-M. and Zhang, C.-H. (2001). Least squares estimators of the mode of a unimodal regression function. *Ann. Statist.* **29** 648–665. MR1865335
[26] Silverman, B.W. (1985). Some aspects of the spline smoothing approach to nonparametric regression curve fitting (with discussion). *J. Roy. Statist. Soc. Ser. B* **47** 1–52. MR0805063
[27] Szabó, B., van der Vaart, A.W. and van Zanten, J.H. (2015). Frequentist coverage of adaptive nonparametric Bayesian credible sets. *Ann. Statist.* **43** 1391–1428. MR3357861
[28] Tang, R., Banerjee, M. and Michailidis, G. (2011). A two-stage hybrid procedure for estimating an inverse regression function. *Ann. Statist.* **39** 956–989. MR2816344
[29] Tsybakov, A.B. (1990). Recursive estimation of the mode of a multivariate distribution. *Probl. Inf. Transm.* **26** 31–37. MR1051586
[30] van der Vaart, A.W. (1998). *Asymptotic Statistics*. Cambridge: Cambridge Univ. Press. MR1652247
[31] van der Vaart, A.W. and van Zanten, J.H. (2008). Reproducing kernel Hilbert spaces of Gaussian priors. In *Pushing the Limits of Contemporary Statistics: Contributions in Honor of Jayanta K. Ghosh. Inst. Math. Stat. (IMS) Collect.* **3** 200–222. Beachwood, OH: IMS. MR2459226
[32] van der Vaart, A.W. and Wellner, J.A. (1996). *Weak Convergence and Empirical Processes With Applications to Statistics*. New York: Springer. MR1385671

[33] Yoo, W.W. and Ghosal, S. (2016). Supremum norm posterior contraction and credible sets for nonparametric multivariate regression. *Ann. Statist.* **44** 1069–1102. MR3485954
[34] Yoo, W.W. and Ghosal, S. (2018). Supplement to "Bayesian mode and maximum estimation and accelerated rates of contraction." DOI:10.3150/18-BEJ1056SUPP.
[35] Yoo, W.W., Rivoirard, V. and Rousseau, J. (2018). Adaptive supremum norm posterior contraction: Wavelet spike-and-slab and anisotropic Besov spaces. arXiv:1708.01909 [math.ST].
[36] Yoo, W.W. and van der Vaart, A.W. (2018). The Bayes Lepski's method and credible bands through volume of tubular neighborhoods. arXiv:1711.06926 [math.ST].

*Bernoulli* **25**(3), 2019, 2359–2408
https://doi.org/10.3150/18-BEJ1057

# Bootstrapping INAR models

CARSTEN JENTSCH[1] and CHRISTIAN H. WEISS[2]

[1]*Technische Universität Dortmund, Fakultät Statistik, D-44221 Dortmund, Germany. E-mail: jentsch@statistik.tu-dortmund.de*
[2]*Helmut Schmidt University, Department of Mathematics and Statistics, PO Box 700822, D-22008 Hamburg, Germany. E-mail: weissc@hsu-hh.de*

Integer-valued autoregressive (INAR) models form a very useful class of processes to deal with time series of counts. Statistical inference in these models is commonly based on asymptotic theory, which is available only under additional parametric conditions and further restrictions on the model order. For general INAR models, such results are not available and might be cumbersome to derive. Hence, we investigate how the INAR model structure and, in particular, its similarity to classical autoregressive (AR) processes can be exploited to develop an asymptotically valid bootstrap procedure for INAR models. Although, in a common formulation, INAR models share the autocorrelation structure with AR models, it turns out that (a) consistent estimation of the INAR coefficients is not sufficient to compute proper 'INAR residuals' to formulate a valid model-based bootstrap scheme, and (b) a naïve application of an AR bootstrap will generally fail. Instead, we propose a general INAR-type bootstrap procedure and discuss parametric as well as semi-parametric implementations. The latter approach is based on a joint semi-parametric estimator of the INAR coefficients and the innovations' distribution. Under mild regularity conditions, we prove bootstrap consistency of our procedure for statistics belonging to the class of functions of generalized means. In an extensive simulation study, we provide numerical evidence of our theoretical findings and illustrate the superiority of the proposed INAR bootstrap over some obvious competitors. We illustrate our method by an application to a real data set about iceberg orders for the Lufthansa stock.

*Keywords:* bootstrap consistency; functions of generalized means; INAR residuals; parametric bootstrap; semi-parametric bootstrap; semi-parametric estimation; time series of counts

## References

[1] Al-Osh, M.A. and Alzaid, A.A. (1987). First-order integer-valued autoregressive (INAR(1)) process. *J. Time Ser. Anal.* **8** 261–275. MR0903755
[2] Alzaid, A.A. and Al-Osh, M. (1990). An integer-valued $p$th-order autoregressive structure (INAR($p$)) process. *J. Appl. Probab.* **27** 314–324. MR1052303
[3] Basawa, I.V., Green, T.A., McCormick, W.P. and Taylor, R.L. (1990). Asymptotic bootstrap validity for finite Markov chains. *Comm. Statist. Theory Methods* **19** 1493–1510. MR1073045
[4] Bisaglia, L.B. and Gerolimetto, M. (2016). Estimation of INAR($p$) models using bootstrap. Working Paper Series 06/2016, Univ. Padova.
[5] Brockwell, P.J. and Davis, R.A. (1991). *Time Series: Theory and Methods*, 2nd ed. *Springer Series in Statistics*. New York: Springer. MR1093459
[6] Bu, R., McCabe, B. and Hadri, K. (2008). Maximum likelihood estimation of higher-order integer-valued autoregressive processes. *J. Time Ser. Anal.* **29** 973–994. MR2464949
[7] Bühlmann, P. (1995). Bootstraps for time series. Technical Report 431, Dept. Statistics, Univ. California, Berkeley.

1350-7265 

[8] Bühlmann, P. (1997). Sieve bootstrap for time series. *Bernoulli* **3** 123–148. MR1466304
[9] Cardinal, M., Roy, R. and Lambert, J. (1999). On the application of integer-valued time series models for the analysis of disease incidence. *Stat*. *Med*. **18** 2025–2039.
[10] De Schepper, A. and Heijnen, B. (1995). General restrictions on tail probabilities. *J*. *Comput*. *Appl*. *Math*. **64** 177–188. MR1364980
[11] Doukhan, P., Fokianos, K. and Li, X. (2012). On weak dependence conditions: The case of discrete valued processes. *Statist*. *Probab*. *Lett*. **82** 1941–1948. MR2970296
[12] Doukhan, P., Fokianos, K. and Li, X. (2013). Corrigendum to "On weak dependence conditions: The case of discrete valued processes" [Statist. Probab. Lett. 82 (2012) 1941–1948] [MR2970296]. *Statist*. *Probab*. *Lett*. **83** 674–675. MR3007006
[13] Drost, F.C., van den Akker, R. and Werker, B.J.M. (2009). Efficient estimation of auto-regression parameters and innovation distributions for semiparametric integer-valued AR($p$) models. *J*. *R*. *Stat*. *Soc*. *Ser*. *B*. *Stat*. *Methodol*. **71** 467–485. MR2649605
[14] Du, J.G. and Li, Y. (1991). The integer-valued autoregressive (INAR($p$)) model. *J*. *Time Ser*. *Anal*. **12** 129–142. MR1108796
[15] Efron, B. (1979). Bootstrap methods: Another look at the jackknife. *Ann*. *Statist*. **7** 1–26. MR0515681
[16] Fink, T. and Kreiß, J.-P. (2013). Bootstrap for random coefficient autoregressive models. *J*. *Time Series Anal*. **34** 646–667. MR3127212
[17] Freeland, R.K. and McCabe, B. (2005). Asymptotic properties of CLS estimators in the Poisson AR(1) model. *Statist*. *Probab*. *Lett*. **73** 147–153. MR2159250
[18] Freeland, R.K. and McCabe, B.P.M. (2004). Analysis of low count time series data by Poisson autoregression. *J*. *Time Ser*. *Anal*. **25** 701–722. MR2089191
[19] Ibragimov, I.A. (1962). Some limit theorems for stationary processes. *Theory Probab*. *Appl*. **7** 349–382.
[20] Jazi, M.A., Jones, G. and Lai, C.-D. (2012). First-order integer valued AR processes with zero inflated Poisson innovations. *J*. *Time Series Anal*. **33** 954–963. MR2991911
[21] Jentsch, C. and Leucht, A. (2016). Bootstrapping sample quantiles of discrete data. *Ann*. *Inst*. *Statist*. *Math*. **68** 491–539. MR3489740
[22] Jentsch, C. and Politis, D.N. (2013). Valid resampling of higher-order statistics using the linear process bootstrap and autoregressive sieve bootstrap. *Comm*. *Statist*. *Theory Methods* **42** 1277–1293. MR3031281
[23] Jentsch, C. and Weiß, C.H. (2019). Supplement to "Bootstrapping INAR models." DOI:10.3150/18-BEJ1057SUPP.
[24] Johnson, N.L., Kemp, A.W. and Kotz, S. (2005). *Univariate Discrete Distributions*, 3rd ed. Hoboken, NJ: John Wiley & Sons. MR2163227
[25] Jung, R.C., McCabe, B.P.M. and Tremayne, A.R. (2016). Model validation and diagnostics. In *Handbook of Discrete-Valued Time Series*. *Chapman & Hall/CRC Handb*. *Mod*. *Stat*. *Methods* 189–218. Boca Raton, FL: CRC Press. MR3699406
[26] Jung, R.C. and Tremayne, A.R. (2006). Coherent forecasting in integer time series models. *Int*. *J*. *Forecast*. **22** 223–238.
[27] Jung, R.C. and Tremayne, A.R. (2011). Convolution-closed models for count time series with applications. *J*. *Time Series Anal*. **32** 268–280. MR2808254
[28] Kim, H.-Y. and Park, Y. (2006). Bootstrap confidence intervals for the INAR($p$) process. *Korean Commun*. *Stat*. **13** 343–358.
[29] Kim, H.-Y. and Park, Y. (2008). A non-stationary integer-valued autoregressive model. *Statist*. *Papers* **49** 485–502. MR2399216
[30] Kreiß, J.-P. (1988). Asymptotical inference for a class of stochastic processes. Habilitationsschrift, Univ. Hamburg.

[31] Kreiß, J.-P. (1992). Bootstrap procedures for AR($\infty$)-processes. In *Bootstrapping and Related Techniques* (*Trier*, 1990). *Lecture Notes in Econom. and Math. Systems* **376** 107–113. Berlin: Springer. MR1238505
[32] Kreiß, J.-P. (1997). Asymptotical properties of residual bootstrap for autoregression. Preprint, TU Braunschweig.
[33] Kreiß, J.-P. and Paparoditis, E. (2011). Bootstrap methods for dependent data: A review. *J. Korean Statist. Soc.* **40** 357–378. MR2906623
[34] Kreiß, J.-P., Paparoditis, E. and Politis, D.N. (2011). On the range of validity of the autoregressive sieve bootstrap. *Ann. Statist.* **39** 2103–2130. MR2893863
[35] Künsch, H.R. (1989). The jackknife and the bootstrap for general stationary observations. *Ann. Statist.* **17** 1217–1241. MR1015147
[36] Latour, A. (1998). Existence and stochastic structure of a non-negative integer-valued autoregressive process. *J. Time Ser. Anal.* **19** 439–455. MR1652193
[37] McKenzie, E. (1985). Some simple models for discrete variate time series. *Water Resour. Bull.* **21** 645–650.
[38] Meintanis, S.G. and Karlis, D. (2014). Validation tests for the innovation distribution in INAR time series models. *Comput. Statist.* **29** 1221–1241. MR3266056
[39] Meyer, M., Jentsch, C. and Kreiß, J.-P. (2015). Baxter's inequality and sieve bootstrap for random fields. Working Paper, Univ. Mannheim.
[40] Meyer, M., Jentsch, C. and Kreiß, J.-P. (2017). Baxter's inequality and sieve bootstrap for random fields. *Bernoulli* **23** 2988–3020. MR3654797
[41] Meyer, M. and Kreiß, J.-P. (2015). On the vector autoregressive sieve bootstrap. *J. Time Series Anal.* **36** 377–397. MR3343007
[42] Park, Y. and Kim, H.-Y. (2012). Diagnostic checks for integer-valued autoregressive models using expected residuals. *Statist. Papers* **53** 951–970. MR2992927
[43] Patton, A., Politis, D.N. and White, H. (2009). Correction to "Automatic block-length selection for the dependent bootstrap" by D. Politis and H. White [MR2041534]. *Econometric Rev.* **28** 372–375. MR2532278
[44] Pavlopoulos, H. and Karlis, D. (2008). INAR(1) modeling of overdispersed count series with an environmental application. *Environmetrics* **19** 369–393. MR2440038
[45] Politis, D.N. and White, H. (2004). Automatic block-length selection for the dependent bootstrap. *Econometric Rev.* **23** 53–70. MR2041534
[46] Schweer, S. (2016). A goodness-of-fit test for integer-valued autoregressive processes. *J. Time Series Anal.* **37** 77–98. MR3439533
[47] Schweer, S. and Weiß, C.H. (2014). Compound Poisson INAR(1) processes: Stochastic properties and testing for overdispersion. *Comput. Statist. Data Anal.* **77** 267–284. MR3210062
[48] Silva, I. and Silva, M.E. (2006). Asymptotic distribution of the Yule–Walker estimator for INAR($p$) processes. *Statist. Probab. Lett.* **76** 1655–1663. MR2248854
[49] Steutel, F.W. and van Harn, K. (1979). Discrete analogues of self-decomposability and stability. *Ann. Probab.* **7** 893–899. MR0542141
[50] Tsay, R.S. (1992). Model checking via parametric bootstraps in time series analysis. *Appl. Stat.* **41** 1–15.
[51] Weiß, C.H. (2012). Process capability analysis for serially dependent processes of Poisson counts. *J. Stat. Comput. Simul.* **82** 383–404. MR2897647
[52] Weiß, C.H. (2018). *An Introduction to Discrete-Valued Time Series*. Chichester: Wiley.
[53] Weiß, C.H., Puig, P. and Homburg, A. (2016). Testing for zero inflation in INAR(1) models. *Statist. Papers*. To appear. DOI:10.1007/s00362-016-0851-y.

[54] Weiß, C.H. and Schweer, S. (2016). Bias corrections for moment estimators in Poisson INAR(1) and INARCH(1) processes. *Statist*. *Probab*. *Lett*. **112** 124–130. MR3475496
[55] Yokoyama, R. (1980). Moment bounds for stationary mixing sequences. *Z*. *Wahrsch*. *Verw*. *Gebiete* **52** 45–57. MR0568258

# Bernoulli Forthcoming Papers

THAI, M.-N.
Birth and death process in mean field type interaction

LEI, T.
Scaling limit of random forests with prescribed degree sequences

ICHIBA, T. and SARANTSEV, A.
Stationary distributions and convergence for Walsh diffusions

LIMIC, V.
The eternal multiplicative coalescent encoding via excursions of Lévy-type processes

BUHL, S., DAVIS, R.A., KLÜPPELBERG, C. and STEINKOHL, C.
Semiparametric estimation for isotropic max-stable space-time processes

GÖTZE, F., NAUMOV, A., SPOKOINY, V. and ULYANOV, V.
Large ball probabilities, Gaussian comparison and anti-concentration

FANG, X., PENG, S., SHAO, Q.-M. and SONG, Y.
Limit theorems with rate of convergence under sublinear expectations

REISS, M. and WAHL, M.
Functional estimation and hypothesis testing in nonparametric boundary models

WEED, J. and BACH, F.
Sharp asymptotic and finite-sample rates of convergence of empirical measures in Wasserstein distance

MARGUET, A.
Uniform sampling in a structured branching population

ABRAHAM, K.
Nonparametric Bayesian posterior contraction rates for scalar diffusions with high-frequency data

HUESMANN, M. and TREVISAN, D.
A Benamou–Brenier formulation of martingale optimal transport

BHATTACHARJEE, C. and GOLDSTEIN, L.
Dickman approximation in simulation, summations and perpetuities

FAN, X., GRAMA, I., LIU, Q. and SHAO, Q.-M.
Self-normalized Cramér type moderate deviations for martingales

RAIČ, M.
A multivariate Berry–Esséen theorem with explicit constants

DURMUS, A. and MOULINES, É.
High-dimensional Bayesian inference via the unadjusted Langevin algorithm

BECT, J., BACHOC, F. and GINSBOURGER, D.
A supermartingale approach to Gaussian process based sequential design of experiments

*Continues*

# Bernoulli Forthcoming Papers—*Continued*

ANH, V., LEONENKO, N., OLENKO, A. and VASKOVYCH, V.
On rate of convergence in non-central limit theorems

LIU, Z., BLANCHET, J.H., DIEKER, A.B. and MIKOSCH, T.
On logarithmically optimal exact simulation of max-stable and related random fields on a compact set

DOLERA, E. and REGAZZINI, E.
Uniform rates of the Glivenko–Cantelli convergence and their use in approximating Bayesian inferences

BELLEC, P. C.
Localized Gaussian width of $M$-convex hulls with applications to Lasso and convex aggregation

DITZHAUS, M.
Signal detection via Phi-divergences for general mixtures

BALAN, R.M. and SONG, J.
Second order Lyapunov exponents for parabolic and hyperbolic Anderson models

NGUYEN, V.H.
$\Phi$-entropy inequalities and asymmetric covariance estimates for convex measures

LIVINGSTONE, S., BETANCOURT, M., BYRNE, S. and GIROLAMI, M.
On the geometric ergodicity of Hamiltonian Monte Carlo

ALONSO-GUTIÉRREZ, D., PROCHNO, J. and THÄLE, C.
Gaussian fluctuations for high-dimensional random projections of $\ell_p^n$-balls

CHZHEN, E., HEBIRI, M. and SALMON, J.
On Lasso refitting strategies

MERLEVÈDE, F., PELIGRAD, M. and UTEV, S.
Functional CLT for martingale-like nonstationary dependent structures

VARVENNE, M.
Rate of convergence to equilibrium for discrete-time stochastic dynamics with memory

BALABDAOUI, F., DUROT, C. and JANKOWSKI, H.
Least squares estimation in the monotone single index model

HONDA, T., ING, C.-K. and WU, W.-Y.
Adaptively weighted group Lasso for semiparametric quantile regression models

ALETTI, G., CRIMALDI, I. and GHIGLIETTI, A.
Networks of reinforced stochastic processes: Asymptotics for the empirical means

BUTLER, R.W. and WOOD, A.T.A.
Limiting saddlepoint relative errors in large deviation regions under purely Tauberian conditions

*Continues*

# Bernoulli Forthcoming Papers—*Continued*

JIANG, W. and ZHANG, C.-H.
Rate of divergence of the nonparametric likelihood ratio test for Gaussian mixtures

BLANCHARD, G. and ZADOROZHNYI, O.
Concentration of weakly dependent Banach-valued sums and applications to statistical learning methods

GREENSHTEIN, E., MANSURA, A. and RITOV, Y.
Nonparametric empirical Bayes improvement of shrinkage estimators with applications to time series

MALLEIN, B. and RAMASSAMY, S.
Two-sided infinite-bin models and analyticity for Barak–Erdős graphs

CÉNAC, P., DE LOYNES, B., OFFRET, Y. and ROUSSELLE, A.
Recurrence of multidimensional persistent random walks. Fourier and series criteria

PILAVAKIS, D., PAPARODITIS, E. and SAPATINAS, T.
Moving block and tapered block bootstrap for functional time series with an application to the $K$-sample mean problem

FAN, J.Y., HAMZA, K., JAGERS, P. and KLEBANER, F.
Convergence of the age structure of general schemes of population processes

BERTAIL, P. and CLÉMENÇON, S.
Bernstein-type exponential inequalities in survey sampling: Conditional Poisson sampling schemes

BARTHELMÉ, S., AMBLARD, P.-O. and TREMBLAY, N.
Asymptotic equivalence of fixed-size and varying-size determinantal point processes

HEINY, J. and MIKOSCH, T.
The eigenstructure of the sample covariance matrices of high-dimensional stochastic volatility models with heavy tails

PITMAN, J. and YAKUBOVICH, Y.
Gaps and interleaving of point processes in sampling from a residual allocation model

LIN, S.
Harmonic measure for biased random walk in a supercritical Galton–Watson tree

FRANCESCHI, S. and RASCHEL, K.
Integral expression for the stationary distribution of reflected Brownian motion in a wedge

GOUÉRÉ, J.-B. and THÉRET, M.
Equivalence of some subcritical properties in continuum percolation

RAVNER, L., BOXMA, O. and MANDJES, M.
Estimating the input of a Lévy-driven queue by Poisson sampling of the workload process

*Continues*

# Bernoulli Forthcoming Papers—*Continued*

COLLING, B. and VAN KEILEGOM, I.
Estimation of fully nonparametric transformation models

SHIOZAWA, Y. and WANG, J.
Long-time heat kernel estimates and upper rate functions of Brownian motion type for symmetric jump processes

CHAKRABORTY, S. and KHARE, K.
Consistent estimation of the spectrum of trace class Data Augmentation algorithms

KOKOSZKA, P., STOEV, S. and XIONG, Q.
Principal components analysis of regularly varying functions

KLOPP, O., LU, Y., TSYBAKOV, A.B. and ZHOU, H.H.
Structured matrix estimation and completion

BERTAIL, P. and PORTIER, F.
Rademacher complexity for Markov chains: Applications to kernel smoothing and Metropolis–Hastings

BURDZY, K., KOŁODZIEJEK, B. and TADIĆ, T.
Inverse exponential decay: Stochastic fixed point equation and ARMA models

SAUMARD, A.
Weighted Poincaré inequalities, concentration inequalities and tail bounds related to Stein kernels in dimension one

GÖTZE, F. and SAMBALE, H.
Second order concentration via logarithmic Sobolev inequalities

CASTILLO, I. and SZABÓ, B.
Spike and slab empirical Bayes sparse credible sets

GEISS, C., LABART, C. and LUOTO, A.
Random walk approximation of BSDEs with Hölder continuous terminal condition

HIRSCH, C. and MÖNCH, C.
Distances and large deviations in the spatial preferential attachment model

BELITSER, E. and NURUSHEV, N.
Needles and straw in a haystack: Robust confidence for possibly sparse sequences

BONNET, G. and CHENAVIER, N.
The maximal degree in a Poisson–Delaunay graph

MAZUR, S., OTRYAKHIN, D. and PODOLSKIJ, M.
Estimation of the linear fractional stable motion

BOGERD, K., CASTRO, R.M. and VAN DER HOFSTAD, R.
Cliques in rank-1 random graphs: The role of inhomogeneity

MATZINGER, H. and HAUSER, R.
Microscopic Path Structure of Optimally Aligned Random Sequences

*Continues*

# Bernoulli Forthcoming Papers—*Continued*

CHENG, D., CAMMAROTA, V., FANTAYE, Y., MARRINUCCI, D. and SCHWARTZMAN, A.
Multiple testing of local maxima for detection of peaks on the (celestial) sphere

BERHARD, H. and DAS, B.
Heavy-tailed random walks, buffered queues and hidden large deviations

DAVIS, R.A., NIELSEN, M.S. and ROHDE, V.
Stochastic differential equations with a fractionally filtered delay: a semimartingale model for long-range dependent processes

HO, N., NGUYEN, X. and RITOV, Y.
Robust estimation of mixing measures in finite mixture models

NOURDIN, I., PECCATI, G. and SEURET, S.
Sojourn Time Dimensions of Fractional Brownian Motion

GAYNANOVA, I.
Prediction and estimation consistency of sparse multi-class penalized optimal scoring

ZWINGMANN, T. and HOLZMANN, H.
Weak convergence of quantile and expectile processes under general assumptions

GIORDANO, L.M., JOLIS, M. and QUER-SARDANYONS, L.
SPDEs with fractional noise in space: continuity in law with respect to the Hurst index

DING, X.
High dimensional deformed rectangular matrices with applications in matrix denoising

SHI, C. and TANG, B.
Construction Results for Strong Orthogonal Arrays of Strength Three

MA, Z. and LI, X.
Subspace Perspective on Canonical Correlation Analysis: Dimension Reduction and Minimax Rates