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Aims and Scope
BERNOULLI is the journal of the Bernoulli Society for Mathematical Statistics and Probability, issued four times per year. The journal provides a comprehensive account of important developments in the fields of statistics and probability, offering an international forum for both theoretical and applied work.

Bernoulli Society for Mathematical Statistics and Probability
The Bernoulli Society was founded in 1973. It is an autonomous Association of the International Statistical Institute, ISI. According to its statutes, the object of the Bernoulli Society is the advancement, through international contacts, of the sciences of probability (including the theory of stochastic processes) and mathematical statistics and of their applications to all those aspects of human endeavour which are directed towards the increase of natural knowledge and the welfare of mankind.

Meetings: http://www.bernoulli-society.org/index.php/meetings
The Society holds a World Congress every four years; more frequent meetings, coordinated by the Society's standing committees and often organised in collaboration with other organisations, are the European Meeting of Statisticians, the Conference on Stochastic Processes and their Applications, the CLAPEM meeting (Latin-American Congress on Probability and Mathematical Statistics), the European Young Statisticians Meeting, and various meetings on special topics – in the physical sciences in particular. The Society, as an association of the ISI, also collaborates with other ISI associations in the organization of the biennial ISI World Statistics Congresses (formerly ISI Sessions).

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The Society is headed by an Executive Committee. As of February 2020 the Executive Committee consists of: President: Claudia Klüppelberg (Germany); President Elect: Adam Jakubowski (Poland); Past President: Susan Murphy (USA); Treasurer: Geoffrey Grimmett (UK); Scientific Secretary: Song Xi Chen (China); Membership Secretary: Sebastian Engelke (Switzerland); Publicity Secretary: Leonardo Rolla (Argentina); Publication Secretary: Herold Dehling (Germany); ISI Director: Ada van Krimpen (Netherlands). Further, the Society has a twelve member Council and a number of standing committees to carry out the tasks outlined above. Final authority is the general assembly of members of the Society, meeting at least biennially at the ISI World Statistics Congresses.

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Stochastic differential equations with a fractionally filtered delay: A semimartingale model for long-range dependent processes

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In this paper, we introduce a model, the stochastic fractional delay differential equation (SFDDE), which is based on the linear stochastic delay differential equation and produces stationary processes with hyperbolically decaying autocovariance functions. The model departs from the usual way of incorporating this type of long-range dependence into a short-memory model as it is obtained by applying a fractional filter to the drift term rather than to the noise term. The advantages of this approach are that the corresponding long-range dependent solutions are semimartingales and the local behavior of the sample paths is unaffected by the degree of long memory. We prove existence and uniqueness of solutions to the SFDDEs and study their spectral densities and autocovariance functions. Moreover, we define a subclass of SFDDEs which we study in detail and relate to the well-known fractionally integrated CARMA processes. Finally, we consider the task of simulating from the defining SFDDEs.

Keywords: long-range dependence; moving average processes; semimartingales; stochastic differential equations

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Robust estimation of mixing measures in finite mixture models

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In finite mixture models, apart from underlying mixing measure, true kernel density function of each subpopulation in the data is, in many scenarios, unknown. Perhaps the most popular approach is to choose some kernel functions that we empirically believe our data are generated from and use these kernels to fit our models. Nevertheless, as long as the chosen kernel and the true kernel are different, statistical inference of mixing measure under this setting will be highly unstable. To overcome this challenge, we propose flexible and efficient robust estimators of the mixing measure in these models, which are inspired by the idea of minimum Hellinger distance estimator, model selection criteria, and superefficiency phenomenon. We demonstrate that our estimators consistently recover the true number of components and achieve the optimal convergence rates of parameter estimation under both the well- and misspecified kernel settings for any fixed bandwidth. These desirable asymptotic properties are illustrated via careful simulation studies with both synthetic and real data.

Keywords: convergence rates; Fisher singularities; minimum distance estimator; mixture models; model misspecification; model selection; strong identifiability; superefficiency; Wasserstein distances

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Recurrence of multidimensional persistent random walks. Fourier and series criteria

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The recurrence and transience of persistent random walks built from variable length Markov chains are investigated. It turns out that these stochastic processes can be seen as Lévy walks for which the persistence times depend on some internal Markov chain: they admit Markov random walk skeletons. A recurrence versus transience dichotomy is highlighted. Assuming the positive recurrence of the driving chain, a sufficient Fourier criterion for the recurrence, close to the usual Chung–Fuchs one, is given and a series criterion is derived. The key tool is the Nagaev–Guivarc'h method. Finally, we focus on particular two-dimensional persistent random walks, including directionally reinforced random walks, for which necessary and sufficient Fourier and series criteria are obtained. Inspired by (*Adv. Math.* **208** (2007) 680–698), we produce a genuine counterexample to the conjecture of (*Adv. Math.* **117** (1996) 239–252). As for the one-dimensional case studied in (*J. Theoret. Probab.* **31** (2018) 232–243), it is easier for a persistent random walk than its skeleton to be recurrent. However, such examples are much more difficult to exhibit in the higher dimensional context. These results are based on a surprisingly novel – to our knowledge – upper bound for the Lévy concentration function associated with symmetric distributions.

Keywords: concentration functions; Fourier and series recurrence criteria; Fourier perturbations; Markov operators; Markov random walks; persistent random walks; variable length Markov chain

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Convergence of the age structure of general schemes of population processes

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We consider a family of general branching processes with reproduction parameters depending on the age of the individual as well as the population age structure and a parameter K, which may represent the carrying capacity. These processes are Markovian in the age structure. In a previous paper (*Proc. Steklov Inst. Math.* **282** (2013) 90–105), the Law of Large Numbers as $K \to \infty$ was derived. Here we prove the central limit theorem, namely the weak convergence of the fluctuation processes in an appropriate Skorokhod space. We also show that the limit is driven by a stochastic partial differential equation.

Keywords: age-structure dependent population processes; carrying capacity; central limit theorem

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Distances and large deviations in the spatial preferential attachment model

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This paper considers two asymptotic properties of a spatial preferential-attachment model introduced by E. Jacob and P. Mörters (In *Algorithms and Models for the Web Graph* (2013) 14–25 Springer). First, in a regime of strong linear reinforcement, we show that typical distances are at most of doubly-logarithmic order. Second, we derive a large deviation principle for the empirical neighbourhood structure and express the rate function as solution to an entropy minimisation problem in the space of stationary marked point processes.

Keywords: distances; large deviation principle; Poisson point process; preferential attachment

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The maximal degree in a Poisson–Delaunay graph

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We investigate the maximal degree in a Poisson–Delaunay graph in $\mathbf{R}^d$, $d \geq 2$, over all nodes in the window $\mathbf{W}_\rho := \rho^{1/d}[0, 1]^d$ as ρ goes to infinity. The exact order of this maximum is provided in any dimension. In the particular setting $d = 2$, we show that this quantity is concentrated on two consecutive integers with high probability. A weaker version of this result is discussed when $d \geq 3$.

Keywords: degree; Delaunay graph; extreme values; Poisson point process

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Stable processes conditioned to hit an interval continuously from the outside

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Conditioning stable Lévy processes on zero probability events recently became a tractable subject since several explicit formulas emerged from a deep analysis using the Lamperti transformations for self-similar Markov processes. In this article, we derive new harmonic functions and use them to explain how to condition stable processes to hit continuously a compact interval from the outside.

Keywords: Markov processes; probabilistic potential theory; stable processes

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Degeneracy in sparse ERGMs with functions of degrees as sufficient statistics

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A sufficient criterion for “non-degeneracy” is given for Exponential Random Graph Models on sparse graphs with sufficient statistics which are functions of the degree sequence. This criterion explains why statistics such as alternating k-star are non-degenerate, whereas subgraph counts are degenerate. It is further shown that this criterion is “almost” tight. Existence of consistent estimates is then proved for non-degenerate Exponential Random Graph Models.

Keywords: degeneracy; ERGM; normalizing constant; sparse graphs

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A unified principled framework for resampling based on pseudo-populations: Asymptotic theory

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In this paper, a class of resampling techniques for finite populations under πps sampling design is introduced. The basic idea on which they rest is a two-step procedure consisting in: (i) constructing a "pseudo-population" on the basis of sample data; (ii) drawing a sample from the predicted population according to an appropriate resampling design. From a logical point of view, this approach is essentially based on the *plug-in* principle by Efron, at the "sampling design level". Theoretical justifications based on large sample theory are provided. New approaches to construct pseudo populations based on various forms of calibrations are proposed. Finally, a simulation study is performed.

Keywords: πps sampling designs; bootstrap; calibration; confidence intervals; finite populations; resampling; variance estimation

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A Bayesian nonparametric approach to log-concave density estimation

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The estimation of a log-concave density on $\mathbb{R}$ is a canonical problem in the area of shape-constrained nonparametric inference. We present a Bayesian nonparametric approach to this problem based on an exponentiated Dirichlet process mixture prior and show that the posterior distribution converges to the log-concave truth at the (near-) minimax rate in Hellinger distance. Our proof proceeds by establishing a general contraction result based on the log-concave maximum likelihood estimator that prevents the need for further metric entropy calculations. We further present computationally more feasible approximations and both an empirical and hierarchical Bayes approach. All priors are illustrated numerically via simulations.

Keywords: convergence rate; density estimation; Dirichlet mixture; log-concavity; nonparametric hypothesis testing; posterior distribution

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Interacting reinforced stochastic processes: Statistical inference based on the weighted empirical means

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This work deals with a system of *interacting reinforced stochastic processes*, where each process $X^j = (X_{n,j})_n$ is located at a vertex j of a finite weighted directed graph, and it can be interpreted as the sequence of "actions" adopted by an agent j of the network. The interaction among the dynamics of these processes depends on the weighted adjacency matrix W associated to the underlying graph: indeed, the probability that an agent j chooses a certain action depends on its personal "inclination" $Z_{n,j}$ and on the inclinations $Z_{n,h}$, with $h \neq j$, of the other agents according to the entries of W. The best known example of reinforced stochastic process is the Pólya urn.

The present paper focuses on the *weighted* empirical means $N_{n,j} = \sum_{k=1}^{n} q_{n,k} X_{k,j}$, since, for example, the current experience is more important than the past one in reinforced learning. Their almost sure synchronization and some central limit theorems in the sense of stable convergence are proven. The new approach with weighted means highlights the key points in proving some recent results for the personal inclinations $Z^j = (Z_{n,j})_n$ and for the empirical means $\overline{X}^j = (\sum_{k=1}^{n} X_{k,j}/n)_n$ given in recent papers (e.g. Aletti, Crimaldi and Ghiglietti (2019), *Ann. Appl. Probab.* **27** (2017) 3787–3844, Crimaldi *et al. Stochastic Process. Appl.* **129** (2019) 70–101). In fact, with a more sophisticated decomposition of the considered processes, we can understand how the different convergence rates of the involved stochastic processes combine. From an application point of view, we provide confidence intervals for the common limit inclination of the agents and a test statistics to make inference on the matrix W, based on the weighted empirical means. In particular, we answer a research question posed in Aletti, Crimaldi and Ghiglietti (2019).

Keywords: asymptotic normality; complex networks; interacting random systems; reinforced learning; reinforced stochastic processes; synchronization; urn models; weighted empirical means

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Robust regression via mutivariate regression depth

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This paper studies robust regression in the settings of Huber's ϵ-contamination models. We consider estimators that are maximizers of multivariate regression depth functions. These estimators are shown to achieve minimax rates in the settings of ϵ-contamination models for various regression problems including nonparametric regression, sparse linear regression, reduced rank regression, etc. We also discuss a general notion of depth function for linear operators that has potential applications in robust functional linear regression.

Keywords: contamination model; data depth; high-dimensional regression; minimax rate; robust statistics

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Characterization of probability distribution convergence in Wasserstein distance by L^p-quantization error function

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We establish conditions to characterize probability measures by their L^p-quantization error functions in both $\mathbb{R}^d$ and Hilbert settings. This characterization is two-fold: static (identity of two distributions) and dynamic (convergence for the L^p-Wasserstein distance). We first propose a criterion on the quantization level N, valid for any norm on $\mathbb{R}^d$ and any order p based on a geometrical approach involving the Voronoï diagram. Then, we prove that in the L^2-case on a (separable) Hilbert space, the condition on the level N can be reduced to $N = 2$, which is optimal. More quantization based characterization cases in dimension 1 and a discussion of the completeness of a distance defined by the quantization error function can be found at the end of this paper.

Keywords: probability distribution characterization; vector quantization; Voronoï diagram; Wasserstein convergence

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Consistent structure estimation of exponential-family random graph models with block structure

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We consider the challenging problem of statistical inference for exponential-family random graph models based on a single observation of a random graph with complex dependence. To facilitate statistical inference, we consider random graphs with additional structure in the form of block structure. We have shown elsewhere that when the block structure is known, it facilitates consistency results for M-estimators of canonical and curved exponential-family random graph models with complex dependence, such as transitivity. In practice, the block structure is known in some applications (e.g., multilevel networks), but is unknown in others. When the block structure is unknown, the first and foremost question is whether it can be recovered with high probability based on a single observation of a random graph with complex dependence. The main consistency results of the paper show that it is possible to do so under weak dependence and smoothness conditions. These results confirm that exponential-family random graph models with block structure constitute a promising direction of statistical network analysis.

Keywords: exponential families; exponential random graph models; network data; random graphs; stochastic block models

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Dynamic linear discriminant analysis in high dimensional space

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High-dimensional data that evolve dynamically feature predominantly in the modern data era. As a partial response to this, recent years have seen increasing emphasis to address the dimensionality challenge. However, the non-static nature of these datasets is largely ignored. This paper addresses both challenges by proposing a novel yet simple dynamic linear programming discriminant (DLPD) rule for binary classification. Different from the usual static linear discriminant analysis, the new method is able to capture the changing distributions of the underlying populations by modeling their means and covariances as smooth functions of covariates of interest. Under an approximate sparse condition, we show that the conditional misclassification rate of the DLPD rule converges to the Bayes risk in probability *uniformly* over the range of the variables used for modeling the dynamics, when the dimensionality is allowed to grow exponentially with the sample size. The minimax lower bound of the estimation of the Bayes risk is also established, implying that the misclassification rate of our proposed rule is minimax-rate optimal. The promising performance of the DLPD rule is illustrated via extensive simulation studies and the analysis of a breast cancer dataset.

Keywords: Bayes rule; discriminant analysis; dynamic linear programming; high-dimensional data; kernel estimation; sparsity

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Strictly weak consensus in the uniform compass model on $\mathbb{Z}$

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We investigate a model for opinion dynamics, where individuals (modeled by vertices of a graph) hold certain abstract opinions. As time progresses, neighboring individuals interact with each other, and this interaction results in a realignment of opinions closer towards each other. This mechanism triggers formation of consensus among the individuals. Our main focus is on *strong consensus* (i.e., global agreement of all individuals) versus *weak consensus* (i.e., local agreement among neighbors). By extending a known model to a more general opinion space, which lacks a "central" opinion acting as a contraction point, we provide an example of an opinion formation process on the one-dimensional lattice $\mathbb{Z}$ with weak consensus but no strong consensus.

Keywords: consensus formation; Deffuant model; interacting particle system; invariant measures; Markov process; opinion dynamics

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Rates of convergence in de Finetti's representation theorem, and Hausdorff moment problem

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Given a sequence $\{X_n\}_{n\ge 1}$ of exchangeable Bernoulli random variables, the celebrated de Finetti representation theorem states that $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{a.s.} Y$ for a suitable random variable $Y:\Omega\to[0,1]$ satisfying $\mathsf{P}[X_1=x_1,\dots,X_n=x_n|Y]=Y^{\sum_{i=1}^n x_i}(1-Y)^{n-\sum_{i=1}^n x_i}$. In this paper, we study the rate of convergence in law of $\frac{1}{n}\sum_{i=1}^n X_i$ to Y under the Kolmogorov distance. After showing that a rate of the type of $1/n^\alpha$ can be obtained for any index $\alpha\in(0,1]$, we find a sufficient condition on the distribution of Y for the achievement of the optimal rate of convergence, that is $1/n$. Besides extending and strengthening recent results under the weaker Wasserstein distance, our main result weakens the regularity hypotheses on Y in the context of the Hausdorff moment problem.

Keywords: de Finetti's law of large numbers; de Finetti's representation theorem; Edgeworth expansions; exchangeability; Hausdorff moment problem; Kolmogorov distance; Wasserstein distance

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A new McKean–Vlasov stochastic interpretation of the parabolic–parabolic Keller–Segel model: The one-dimensional case

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In this paper, we analyze a stochastic interpretation of the one-dimensional parabolic–parabolic Keller–Segel system without cut-off. It involves an original type of McKean–Vlasov interaction kernel. At the particle level, each particle interacts with all the past of each other particle by means of a time integrated functional involving a singular kernel. At the mean-field level studied here, the McKean–Vlasov limit process interacts with all the past time marginals of its probability distribution in a similarly singular way. We prove that the parabolic–parabolic Keller–Segel system in the whole Euclidean space and the corresponding McKean–Vlasov stochastic differential equation are well-posed for any values of the parameters of the model.

Keywords: chemotaxis model; Keller–Segel system; singular McKean–Vlasov non-linear stochastic differential equation

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On stability of traveling wave solutions for integro-differential equations related to branching Markov processes

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The aim of this paper is to prove stability of traveling waves for integro-differential equations connected with branching Markov processes. In other words, the limiting law of the left-most particle of a (time-continuous) branching Markov process with a Lévy non-branching part is demonstrated. The key idea is to approximate the branching Markov process by a branching random walk and apply the result of Aïdékon [*Ann. Probab.* **41** (2013) 1362–1426] on the limiting law of the latter one.

Keywords: Bramson's correction; branching process; integro-differential equation; spatial spread; traveling wave

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Stratonovich stochastic differential equation with irregular coefficients: Girsanov's example revisited

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In this paper, we study the Stratonovich stochastic differential equation $\mathrm{d}X = |X|^\alpha \circ \mathrm{d}B$, $\alpha \in (-1, 1)$, which has been introduced by Cherstvy et al. (*New J. Phys.* **15** (2013) 083039) in the context of analysis of anomalous diffusions in heterogeneous media. We determine its weak and strong solutions, which are homogeneous strong Markov processes spending zero time at 0: for $\alpha \in (0, 1)$, these solutions have the form

$$X_t^\theta = \big((1-\alpha)B_t^\theta\big)^{1/(1-\alpha)},$$

where B^θ is the θ-skew Brownian motion driven by B and starting at $\frac{1}{1-\alpha}(X_0)^{1-\alpha}$, $\theta \in [-1, 1]$, and $(x)^\gamma = |x|^\gamma \operatorname{sign} x$; for $\alpha \in (-1, 0]$, only the case $\theta = 0$ is possible. The central part of the paper consists in the proof of the existence of a quadratic covariation $[f(B^\theta), B]$ for a locally square integrable function f and is based on the time-reversion technique for Markovian diffusions.

Keywords: generalized Itô's formula; Girsanov's example; heterogeneous diffusion process; local time; non-uniqueness; singular stochastic differential equation; skew Brownian motion; Stratonovich integral; time reversion

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The moduli of non-differentiability for Gaussian random fields with stationary increments

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We establish the exact moduli of non-differentiability of Gaussian random fields with stationary increments. As an application of the result, we prove that the uniform Hölder condition for the maximum local times of Gaussian random fields with stationary increments obtained in Xiao (1997) is optimal. These results are applicable to fractional Riesz–Bessel processes and stationary Gaussian random fields in the Matérn and Cauchy classes.

Keywords: Cauchy class; fractional Riesz–Bessel process; Gaussian random field; local time; modulus of non-differentiability; strong local nondeterministism

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Around the entropic Talagrand inequality

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In this article, we study generalization of the classical Talagrand transport-entropy inequality in which the Wasserstein distance is replaced by the entropic transportation cost. This class of inequalities has been introduced in the recent work (*Probab. Theory Related Fields* **174** (2019) 1–47), in connection with the study of Schrödinger bridges. We provide several equivalent characterizations in terms of reverse hypercontractivity for the heat semigroup, contractivity of the Hamilton–Jacobi–Bellman semigroup and dimension-free concentration of measure. Properties such as tensorization and relations to other functional inequalities are also investigated. In particular, we show that the inequalities studied in this article are implied by a Logarithmic Sobolev inequality and imply Talagrand inequality.

Keywords: functional inequalities; large deviations; optimal transport; semigroups

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A characterization of the finiteness of perpetual integrals of Lévy processes

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We derive a criterium for the almost sure finiteness of perpetual integrals of Lévy processes for a class of real functions including all continuous functions and for general one-dimensional Lévy processes that drifts to plus infinity. This generalizes previous work of Döring and Kyprianou, who considered Lévy processes having a local time, leaving the general case as an open problem. It turns out, that the criterium in the general situation simplifies significantly in the situation, where the process has a local time, but we also demonstrate that in general our criterium can not be reduced. This answers an open problem posed in (*J. Theoret. Probab.* **29** (2016) 1192–1198).

Keywords: Lévy processes; perpetual integrals; potential measures

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Limit theorems for long-memory flows on Wiener chaos

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We consider a long-memory stationary process, defined not through a moving average type structure, but by a flow generated by a measure-preserving transform and by a multiple Wiener–Itô integral. The flow is described using a notion of mixing for infinite-measure spaces introduced by Krickeberg (In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 2* (1967) 431–446 Univ. California Press). Depending on the interplay between the spreading rate of the flow and the order of the multiple integral, one can recover known central or non-central limit theorems, and also obtain joint convergence of multiple integrals of different orders.

Keywords: conservative flows; ergodic theory; Fourth Moment Theorem; long memory; long-range dependence

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On the probability distribution of the local times of diagonally operator-self-similar Gaussian fields with stationary increments

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In this paper, we study the local times of vector-valued Gaussian fields that are 'diagonally operator-self-similar' and whose increments are stationary. Denoting the local time of such a Gaussian field around the spatial origin and over the temporal unit hypercube by Z, we show that there exists $\lambda \in (0,1)$ such that under some quite weak conditions, $\lim_{n\to+\infty} \frac{\sqrt[n]{\mathbb{E}(Z^n)}}{n^\lambda}$ and $\lim_{x\to+\infty} \frac{-\log \mathbb{P}(Z>x)}{x^{\frac{1}{\lambda}}}$ both exist and are strictly positive (possibly $+\infty$). Moreover, we show that if the underlying Gaussian field is 'strongly locally nondeterministic', the above limits will be finite as well. These results are then applied to establish similar statements for the intersection local times of diagonally operator-self-similar Gaussian fields with stationary increments.

Keywords: fractional Brownian fields; Gaussian fields; local times; operator-self-similar random fields; probability tail decay

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Reliable clustering of Bernoulli mixture models

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A Bernoulli Mixture Model (BMM) is a finite mixture of random binary vectors with independent dimensions. The problem of clustering BMM data arises in a variety of real-world applications, ranging from population genetics to activity analysis in social networks. In this paper, we analyze the clusterability of BMMs from a theoretical perspective, when the number of clusters is unknown. In particular, we stipulate a set of conditions on the sample complexity and dimension of the model in order to guarantee the Probably Approximately Correct (PAC)-clusterability of a dataset. To the best of our knowledge, these findings are the first non-asymptotic bounds on the sample complexity of learning or clustering BMMs.

Keywords: high-dimensional statistics; mixture model analysis; PAC-learnability; sample complexity

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Random orthogonal matrices and the Cayley transform

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Random orthogonal matrices play an important role in probability and statistics, arising in multivariate analysis, directional statistics, and models of physical systems, among other areas. Calculations involving random orthogonal matrices are complicated by their constrained support. Accordingly, we parametrize the Stiefel and Grassmann manifolds, represented as subsets of orthogonal matrices, in terms of Euclidean parameters using the Cayley transform. We derive the necessary Jacobian terms for change of variables formulas. Given a density defined on the Stiefel or Grassmann manifold, these allow us to specify the corresponding density for the Euclidean parameters, and vice versa. As an application, we present a Markov chain Monte Carlo approach to simulating from distributions on the Stiefel and Grassmann manifolds. Finally, we establish that the Euclidean parameters corresponding to a uniform orthogonal matrix can be approximated asymptotically by independent normals. This result contributes to the growing literature on normal approximations to the entries of random orthogonal matrices or transformations thereof.

Keywords: Gaussian approximation; Grassmann manifold; Jacobian; Markov chain Monte Carlo; Stiefel manifold

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Efficient estimation in single index models through smoothing splines

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We consider estimation and inference in a single index regression model with an unknown but smooth link function. In contrast to the standard approach of using kernels or regression splines, we use smoothing splines to estimate the smooth link function. We develop a method to compute the penalized least squares estimators (PLSEs) of the parametric and the nonparametric components given independent and identically distributed (i.i.d.) data. We prove the consistency and find the rates of convergence of the estimators. We establish asymptotic normality under mild assumption and prove asymptotic efficiency of the parametric component under homoscedastic errors. A finite sample simulation corroborates our asymptotic theory. We also analyze a car mileage data set and a Ozone concentration data set. The identifiability and existence of the PLSEs are also investigated.

Keywords: least favorable submodel; penalized least squares; semiparametric model

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