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Aims and Scope

BERNOULLI is the journal of the Bernoulli Society for Mathematical Statistics and Probability, issued four times per year. The journal provides a comprehensive account of important developments in the fields of statistics and probability, offering an international forum for both theoretical and applied work.

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The Bernoulli Society was founded in 1973. It is an autonomous Association of the International Statistical Institute, ISI. According to its statutes, the object of the Bernoulli Society is the advancement, through international contacts, of the sciences of probability (including the theory of stochastic processes) and mathematical statistics and of their applications to all those aspects of human endeavour which are directed towards the increase of natural knowledge and the welfare of mankind.

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The papers published in Bernoulli are indexed or abstracted in *Current Index to Statistics, Mathematical Reviews, Statistical Theory and Method Abstracts-Zentralblatt (STMA-Z), Thomson Scientific* and *Zentralblatt für Mathematik* (also available on the *MATH via STN* database and *Compact MATH CD-ROM*). A list of forthcoming papers can be found online at http://www.bernoulli-society.org/index.php/publications/bernoulli-journal/bernoulli-journal-papers

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A general frequency domain method for assessing spatial covariance structures

MATTHEW VAN HALA[1,*], SOUTIR BANDYOPADHYAY[2], SOUMENDRA N. LAHIRI[3] and DANIEL J. NORDMAN[1,**]

[1]*Department of Statistics, Iowa State University, Ames, IA 50011, USA.*
E-mail: [*]*matthew.vanhala@gmail.com;* [**]*dnordman@iastate.edu*
[2]*Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, CO 80401, USA.*
E-mail: sbandyopadhyay@mines.edu
[3]*Department of Statistics, North Carolina State University, Raleigh, NC 27695-8203, USA.*
E-mail: snlahiri@ncsu.edu

When examining dependence in spatial data, it can be helpful to formally assess spatial covariance structures that may not be parametrically specified or fully model-based. That is, one may wish to test for general features regarding spatial covariance without presupposing any particular, or potentially restrictive, assumptions about the joint data distribution. Current methods for testing spatial covariance are often intended for specialized inference scenarios, usually with spatial lattice data. We propose instead a general method for estimation and testing of spatial covariance structure, which is valid for a variety of inference problems (including nonparametric hypotheses) *and* applies to a large class of spatial sampling designs with irregular data locations. In this setting, spatial statistics have limiting distributions with complex standard errors depending on the intensity of spatial sampling, the distribution of sampling locations, and the process dependence. The proposed method has the advantage of providing valid inference in the frequency domain without estimation of such standard errors, which are often intractable, and without particular distributional assumptions about the data (e.g., Gaussianity). To illustrate, we develop the method for formally testing isotropy and separability in spatial covariance and consider confidence regions for spatial parameters in variogram model fitting. A broad result is also presented to justify the method for application to other potential problems and general scenarios with testing spatial covariance. The approach uses spatial test statistics, based on an extended version of empirical likelihood, having simple chi-square limits for calibrating tests. We demonstrate the proposed method through several numerical studies.

Keywords: confidence sets; spatial periodogram; spatial testing; spectral moment conditions; stochastic sampling

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Deviation inequalities for random polytopes in arbitrary convex bodies

VICTOR-EMMANUEL BRUNEL

Department of Statistics, CREST-ENSAE, 5 Avenue Le Chatelier, 92120 Palaiseau, France.
E-mail: victor.emmanuel.brunel@ensae.fr

We prove an exponential deviation inequality for the convex hull of a finite sample of i.i.d. random points with a density supported on an arbitrary convex body in $\mathbb{R}^d$, $d \geq 2$. When the density is uniform, our result yields rate optimal upper bounds for all the moments of the missing volume of the convex hull, uniformly over all convex bodies of $\mathbb{R}^d$: We make no restrictions on their volume, location in the space or smoothness of their boundary. For general densities, the only restriction we make is that the density is bounded from above, even though we believe this restriction is not necessary. However, the density can have any decay to zero near the boundary of its support. After extending an identity due to Efron, we also prove upper bounds for the moments of the number of vertices of the random polytope. Surprisingly, these bounds do not depend on the underlying density and we prove that the growth rates that we obtain are tight in a certain sense. Our results are non asymptotic and hold uniformly over all convex bodies.

Keywords: convex body; convex hull; covering number; density support estimation; deviation inequality; random polytope

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Nested covariance determinants and restricted trek separation in Gaussian graphical models

MATHIAS DRTON[1], ELINA ROBEVA[2] and LUCA WEIHS[3]

[1]*Department of Mathematics, Technical University of Munich, Munich, Germany.*
E-mail: mathias.drton@tum.de
[2]*Department of Mathematics, University of British Columbia, Vancouver, BC, Canada.*
E-mail: erobeva@math.ubc.ca
[3]*Department of Statistics University of Washington, Seattle, WA, USA. E-mail:* lucaw@uw.edu

Directed graphical models specify noisy functional relationships among a collection of random variables. In the Gaussian case, each such model corresponds to a semi-algebraic set of positive definite covariance matrices. The set is given via a parametrization, and much work has gone into obtaining an implicit description in terms of polynomial (in)equalities. Implicit descriptions shed light on problems such as parameter identification, model equivalence and constraint-based statistical inference. For models given by directed acyclic graphs, which represent settings where all relevant variables are observed, there is a complete theory: All conditional independence relations can be found via graphical d-separation and are sufficient for an implicit description. The situation is far more complicated, however, when some of the variables are hidden (or in other words, unobserved or latent). We consider models associated to mixed graphs that capture the effects of hidden variables through correlated error terms. The notion of trek separation explains when the covariance matrix in such a model has submatrices of low rank and generalizes d-separation. However, in many cases, such as the infamous Verma graph, the polynomials defining the graphical model are not determinantal, and hence cannot be explained by d-separation or trek-separation. In this paper, we show that these constraints often correspond to the vanishing of nested determinants and can be graphically explained by the (more general) notion of *restricted trek separation*.

Keywords: conditional independence; covariance matrix; graphical model; trek separation; Verma constraint

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High-dimensional general linear hypothesis tests via non-linear spectral shrinkage

HAORAN LI[1], ALEXANDER AUE[2,*] and DEBASHIS PAUL[2,†]

[1]*Department of Statisitcs, Columbia University, 1255 Amsterdam Avenue, New York, NY 10027, USA. E-mail: hl3369@columbia.edu*
[2]*Department of Statistics, University of California at Davis, One Shields Avenue, Davis, CA 95616, USA. E-mail: *aaue@ucdavis.edu; †debpaul@ucdavis.edu*

We are interested in testing general linear hypotheses in a high-dimensional multivariate linear regression model. The framework includes many well-studied problems such as two-sample tests for equality of population means, MANOVA and others as special cases. A family of rotation-invariant tests is proposed that involves a flexible spectral shrinkage scheme applied to the sample error covariance matrix. The asymptotic normality of the test statistic under the null hypothesis is derived in the setting where dimensionality is comparable to sample sizes, assuming the existence of certain moments for the observations. The asymptotic power of the proposed test is studied under various local alternatives. The power characteristics are then utilized to propose a data-driven selection of the spectral shrinkage function. As an illustration of the general theory, we construct a family of tests involving ridge-type regularization and suggest possible extensions to more complex regularizers. A simulation study is carried out to examine the numerical performance of the proposed tests.

Keywords: general linear hypothesis; local alternatives; random matrix theory; ridge shrinkage; spectral shrinkage

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Exact long time behavior of some regime switching stochastic processes

FILIP LINDSKOG* and ABHISHEK PAL MAJUMDER†

Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden.
E-mail: *lindskog@math.su.se; †abhishek_pal@sutd.edu.sg

Regime switching processes have proved to be indispensable in the modeling of various phenomena, allowing model parameters that traditionally were considered to be constant to fluctuate in a Markovian manner in line with empirical findings. We study diffusion processes of Ornstein–Uhlenbeck type where the drift and diffusion coefficients a and b are functions of a Markov process with a stationary distribution π on a countable state space. Exact long time behavior is determined for the three regimes corresponding to the expected drift: $E_\pi a(\cdot) > 0, = 0, < 0$, respectively. Alongside we provide exact time limit results for integrals of form $\int_0^t b^2(X_s)e^{-2\int_s^t a(X_r)\,dr}\,ds$ for the three different regimes. Finally, we demonstrate natural applications of the findings in terms of Cox–Ingersoll–Ross diffusion and deterministic SIS epidemic models in Markovian environments. The time asymptotic behaviors are naturally expressed in terms of solutions to the well-studied fixed-point equation in law $X \stackrel{d}{=} AX + B$ with $X \perp\!\!\!\perp (A, B)$.

Keywords: Cox Ingersoll Ross; long time behavior; Ornstein Uhlenbeck; regime switching; SIS epidemic model

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Fundamental limits of exact support recovery in high dimensions

ZHENG GAO[*] and STILIAN STOEV[†]

Department of Statistics, University of Michigan, 1085 S. University Ave., Ann Arbor, MI, 48105, USA.
E-mail: [*]*gaozheng@umich.edu;* [†]*sstoev@umich.edu*

We study the support recovery problem for a high-dimensional signal observed with additive noise. With suitable parametrization of the signal sparsity and magnitude of its non-zero components, we characterize a phase-transition phenomenon akin to the signal detection problem studied by Ingster in 1998. Specifically, if the signal magnitude is above the so-called *strong classification boundary*, we show that several classes of well-known procedures achieve asymptotically perfect support recovery as the dimension goes to infinity. This is so, for a very broad class of error distributions with light, rapidly varying tails which may have arbitrary dependence. Conversely, if the signal is below the boundary, then for a very broad class of error dependence structures, no thresholding estimators (including ones with data-dependent thresholds) can achieve perfect support recovery. The proofs of these results exploit a certain *concentration of maxima* phenomenon known as relative stability. We provide a complete characterization of the relative stability phenomenon for Gaussian triangular arrays in terms of their correlation structure. The proof uses classic Sudakov–Fernique and Slepian lemma arguments along with a curious application of Ramsey's coloring theorem.

We note that our study of the strong classification boundary is in a finer, point-wise, rather than minimax, sense. We also establish results on the finite-sample Bayes optimality and sub-optimality of thresholding procedures. Consequently, we obtain a minimax-type characterization of the strong classification boundary for errors with log-concave densities.

Keywords: concentration of maxima; high–dimensional inference; Ramsey theory; rapid variation; relative stability; Sudakov–Fernique inequality; support recovery

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Estimating linear and quadratic forms via indirect observations

ANATOLI JUDITSKY[1] and ARKADI NEMIROVSKI[2]

[1] *LJK, Université Grenoble Alpes, 700 Avenue Centrale 38401 Domaine Universitaire de Saint-Martin-d'Hères, France.*
E-mail: anatoli.juditsky@univ-grenoble-alpes.fr
[2] *Georgia Institute of Technology, Atlanta, GA 30332, USA.*
E-mail: nemirovs@isye.gatech.edu

In this paper, we further develop the approach, originating in Juditsky and Nemirovski (*Ann. Statist.* **37** (2009) 2278–2300), to "computation-friendly" statistical estimation via Convex Programming.Our focus is on estimating a linear or quadratic form of an unknown "signal," known to belong to a given convex compact set, via noisy indirect observations of the signal. Classical theoretical results on the subject deal with precisely stated statistical models and aim at designing statistical inferences and quantifying their performance in a closed analytic form. In contrast to this traditional (highly instructive) descriptive framework, the approach we promote here can be qualified as operational – the estimation routines and their risks are not available "in a closed form," but are yielded by an efficient computation. All we know in advance is that under favorable circumstances the risk of the resulting estimate, whether high or low, is provably near-optimal under the circumstances. As a compensation for the lack of "explanatory power," this approach is applicable to a much wider family of observation schemes than those where "closed form descriptive analysis" is possible.

We discuss applications of this approach to classical problems of estimating linear forms of parameters of sub-Gaussian distribution and quadratic forms of parameters of Gaussian and discrete distributions. The performance of the constructed estimates is illustrated by computation experiments in which we compare the risks of the constructed estimates with (numerical) lower bounds for corresponding minimax risks for randomly sampled estimation problems.

Keywords: linear and quadratic functional estimation; linear estimation; statistical linear inverse problems

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Learning the distribution of latent variables in paired comparison models with round-robin scheduling

ROLAND DIEL[1], SYLVAIN LE CORFF[2] and MATTHIEU LERASLE[3]

[1]*Laboratoire J.A.Dieudonné, UMR CNRS-UNS 6621, Université de Nice Sophia-Antipolis, Nice, France.*
E-mail: *roland.diel@univ-cotedazur.fr*
[2]*Samovar, Télécom SudParis, Département CITI, TIPIC, Institut Polytechnique de Paris, Palaiseau, France.*
E-mail: *sylvain.le_corff@telecom-sudparis.eu*
[3]*CNRS, ENSAE, CREST, Institut Polytechnique de Paris, Palaiseau, France.*
E-mail: *matthieu.lerasle@ensae.fr*

Paired comparison data considered in this paper originate from the comparison of a large number N of individuals in couples. The dataset is a collection of results of contests between two individuals when each of them has faced n opponents, where $n \ll N$. Individuals are represented by independent and identically distributed random parameters characterizing their abilities. The paper studies the maximum likelihood estimator of the parameters distribution. The analysis relies on the construction of a graphical model encoding conditional dependencies of the observations which are the outcomes of the first n contests each individual is involved in. This graphical model allows to prove geometric loss of memory properties and deduce the asymptotic behavior of the likelihood function. This paper sets the focus on graphical models obtained from round-robin scheduling of these contests. Following a classical construction in learning theory, the asymptotic likelihood is used to measure performance of the maximum likelihood estimator. Risk bounds for this estimator are finally obtained by sub-Gaussian deviation results for Markov chains applied to the graphical model.

Keywords: latent variables; nonasymptotic risk bounds; nonparametric estimation; paired comparisons data

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Fractional stochastic wave equation driven by a Gaussian noise rough in space

JIAN SONG[1,2], XIAOMING SONG[3] and FANGJUN XU[4,5]

[1]*Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao, Shandong, 266237, China*
[2]*School of Mathematics, Shandong University, Jinan, Shandong 250100, China.*
E-mail: txjsong@hotmail.com
[3]*Department of Mathematics, Drexel University, Philadelphia, PA 19104, USA.*
E-mail: xs73@drexel.edu
[4]*Key Laboratory of Advanced Theory and Application in Statistics and Data Science – MOE, School of Statistics, East China Normal University, Shanghai 200062, China*
[5]*NYU–ECNU Institute of Mathematical Sciences, NYU Shanghai, Shanghai 200062, China.*
E-mail: fangjunxu@gmail.com

In this article, we consider fractional stochastic wave equations on $\mathbb{R}$ driven by a multiplicative Gaussian noise which is white/colored in time and has the covariance of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in space. We prove the existence and uniqueness of the mild Skorohod solution, establish lower and upper bounds for the pth moment of the solution for all $p \geq 2$, and obtain the Hölder continuity in time and space variables for the solution.

Keywords: fractional Brownian motion; Hölder continuity; intermittency; Malliavin calculus; Skorohod integral; stochastic wave equation

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Signature cumulants, ordered partitions, and independence of stochastic processes

PATRIC BONNIER* and HARALD OBERHAUSER†

Mathematical Institute, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, OX2 6GG, Oxford, United Kingdom.
E-mail: *Patric.Bonnier@maths.ox.ac.uk; †Harald.Oberhauser@maths.ox.ac.uk

The sequence of so-called signature moments describes the laws of many stochastic processes in analogy with how the sequence of moments describes the laws of vector-valued random variables. However, even for vector-valued random variables, the sequence of cumulants is much better suited for many tasks than the sequence of moments. This motivates us to study so-called signature cumulants. To do so, we develop an elementary combinatorial approach and show that in the same way that cumulants relate to the lattice of partitions, signature cumulants relate to the lattice of so-called "ordered partitions". We use this to give a new characterisation of independence of multivariate stochastic processes. Finally, we construct a family of unbiased minimum-variance estimators of signature cumulants and show that even for the simple example of a diffusion with constant drift and volatility, such signature cumulant estimators outperform signature moment estimators.

Keywords: cumulants; geometric rough paths; partitions; path signatures; stochastic processes

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Distance covariance for discretized stochastic processes

HEROLD DEHLING[1], MUNEYA MATSUI[2], THOMAS MIKOSCH[3], GENNADY SAMORODNITSKY[4] and LALEH TAFAKORI[5]

[1]*Department of Mathematics, Ruhr-Universität Bochum, 44780 Bochum, Germany.*
E-mail: herold.dehling@rub.de
[2]*Department of Business Administration, Nanzan University, 18 Yamazato-cho, Showa-ku, Nagoya 466-8673, Japan.*
E-mail: mmuneya@gmail.com
[3]*Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark.*
E-mail: mikosch@math.ku.dk
[4]*School of Operations Research and Information Engineering, Cornell University, 220 Rhodes Hall, Ithaca, NY 14853, USA.*
E-mail: gs18@cornell.edu
[5]*Department of Mathematical Sciences, School of Science, RMIT University, Melbourne, VIC, Australia.*
E-mail: laleh.tafakori@rmit.edu.au

Given an i.i.d. sequence of pairs of stochastic processes on the unit interval we construct a measure of independence for the components of the pairs. We define distance covariance and distance correlation based on approximations of the component processes at finitely many discretization points. Assuming that the mesh of the discretization converges to zero as a suitable function of the sample size, we show that the sample distance covariance and correlation converge to limits which are zero if and only if the component processes are independent. To construct a test for independence of the discretized component processes, we show consistency of the bootstrap for the corresponding sample distance covariance/correlation.

Keywords: distance covariance; empirical characteristic function; stochastic process; test of independence

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Rate-optimal nonparametric estimation for random coefficient regression models

HAJO HOLZMANN[1] and ALEXANDER MEISTER[2]

[1]*Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, 35037 Marburg, Germany. E-mail: holzmann@mathematik.uni-marburg.de*
[2]*Institut für Mathematik, Universität Rostock, 18051 Rostock, Germany. E-mail: alexander.meister@uni-rostock.de*

Random coefficient regression models are a popular tool for analyzing unobserved heterogeneity, and have seen renewed interest in the recent econometric literature. In this paper, we obtain the optimal pointwise convergence rate for estimating the density in the linear random coefficient model over Hölder smoothness classes, and in particular show how the tail behavior of the design density impacts this rate. In contrast to previous suggestions, the estimator that we propose and that achieves the optimal convergence rate does not require dividing by a nonparametric density estimate. The optimal choice of the tuning parameters in the estimator depends on the tail parameter of the design density and on the smoothness level of the Hölder class, and we also study adaptive estimation with respect to both parameters.

Keywords: adaptive estimation; ill-posed inverse problem; minimax risk; nonparametric estimation

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Inference for semiparametric Gaussian copula model adjusted for linear regression using residual ranks

YUE ZHAO[1,*], IRÈNE GIJBELS[2] and INGRID VAN KEILEGOM[1,†]

[1]*Research Centre for Operations Research and Statistics (ORSTAT), KU Leuven, Naamsestraat 69, 3000 Leuven, Belgium.*
E-mail: [*]*yue.zhao@york.ac.uk;* [†]*ingrid.vankeilegom@kuleuven.be*
[2]*Department of Mathematics and Leuven Statistics Research Center (LStat), KU Leuven, Celestijnenlaan 200B, B-3001 Leuven (Heverlee), Belgium.*
E-mail: irene.gijbels@kuleuven.be

We investigate the inference of the copula parameter in the semiparametric Gaussian copula model when the copula component, subject to the influence of a covariate, is only indirectly observed as the response in a linear regression model. We consider estimators based on residual ranks instead of the usual but unobservable oracle ranks. We first study two such estimators for the copula correlation matrix, one via inversion of Spearman's rho and the other via normal scores rank correlation estimator. We show that these estimators are asymptotically equivalent to their counterparts based on the oracle ranks. Then, for the copula correlation matrix under constrained parametrizations, we show that the classical one-step estimator in conjunction with the residual ranks remains semiparametrically efficient for estimating the copula parameter. The accuracy of the estimators based on residual ranks is confirmed by simulation studies.

Keywords: U-process; Gaussian copula; normal scores rank correlation estimator; residual rank; semiparametric efficiency; Spearman's rho

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Asymptotic properties of penalized splines for functional data

LUO XIAO

Department of Statistics, North Carolina State University, Raleigh, USA.
E-mail: lxiao5@ncsu.edu

Penalized spline methods are popular for functional data analysis but their asymptotic properties have not been established. We present a theoretic study of the L_2 and uniform convergence of penalized splines for estimating the mean and covariance functions of functional data under general settings. The established convergence rates for the mean function estimation are mini-max rate optimal and the rates for the covariance function estimation are comparable to those using other smoothing methods.

Keywords: L_2 convergence; functional data analysis; nonparametric regression; penalized splines; uniform convergence

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A perturbation analysis of Markov chains models with time-varying parameters

LIONEL TRUQUET

CREST UMR 9194, ENSAI, Campus de Ker-Lann, Rue Blaise Pascal – BP 37203, 35712 BRUZ cedex, France.
E-mail: lionel.truquet@ensai.fr

We study some regularity properties in locally stationary Markov models which are fundamental for controlling the bias of nonparametric kernel estimators. In particular, we provide an alternative to the standard notion of derivative process developed in the literature and that can be used for studying a wide class of Markov processes. To this end, for some families of V-geometrically ergodic Markov kernels indexed by a real parameter u, we give conditions under which the invariant probability distribution is differentiable with respect to u, in the sense of signed measures. Our results also complete the existing literature for the perturbation analysis of Markov chains, in particular when exponential moments are not finite. Our conditions are checked on several original examples of locally stationary processes such as integer-valued autoregressive processes, categorical time series or threshold autoregressive processes.

Keywords: local stationarity; time-inhomogeneous Markov chains

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Testing and inference for fixed times of discontinuity in semimartingales

VIKTOR TODOROV[1]

[1]*Department of Finance, Northwestern University, Evanston, IL 60208, USA.*
E-mail: v-todorov@northwestern.edu

We develop a nonparametric test for deciding whether a semimartingale process, modeling an asset price, contains a fixed time of discontinuity, i.e., a positive probability of a jump, at a given point in time, and we further propose a rate-optimal estimator of the jump distribution when this is the case. Itô semimartingales used commonly in applied work have absolutely continuous in time, with respect to Lebesgue measure, jump compensators, and this rules out fixed times of discontinuity in their paths. However, certain phenomena, such as scheduled economic announcements in finance, make the existence of such discontinuities a possibility. The inference in the paper is based on noisy observations of options written on the asset with different strikes and two different expiration dates. The asymptotics is joint in which the times to maturity of the options shrink to zero and the number of observed options increases to infinity. The test is based on estimates of the characteristic function of the increments of the semimartingale, constructed from the option data, and the fact that the asymptotic limit of the increments and their characteristic functions is different with and without fixed time of discontinuity. The limit distribution of the test statistic is derived and feasible inference is developed on the basis of wild bootstrap type techniques. A Monte Carlo and an empirical illustration show the applicability of the developed inference procedures.

Keywords: bootstrap; fixed time of discontinuity; jumps; nonparametric inference; options; stable convergence; stochastic volatility; time-changed Lévy process

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Penalisation techniques for one-dimensional reflected rough differential equations

ALEXANDRE RICHARD[1], ETIENNE TANRÉ[2] and SOLEDAD TORRES[3]

[1]*Université Paris-Saclay, CentraleSupélec, MICS and CNRS FR-3487, France.*
E-mail: *Alexandre.Richard@centralesupelec.fr*
[2]*Université Côte d'Azur, Inria, France.*
E-mail: *Etienne.Tanre@inria.fr*
[3]*Facultad de Ingeniería, CIMFAV, Universidad de Valparaíso, Casilla 123-V, 4059 Valparaíso, Chile.*
E-mail: *Soledad.Torres@uv.cl*

In this paper, we solve real-valued rough differential equations (RDEs) reflected on an irregular boundary. The solution Y is constructed as the limit of a sequence $(Y^n)_{n\in\mathbb{N}}$ of solutions to RDEs with unbounded drifts $(\psi_n)_{n\in\mathbb{N}}$. The penalisation ψ_n increases with n. Along the way, we thus also provide an existence theorem and a Doss–Sussmann representation for RDEs with a drift growing at most linearly. In addition, a speed of convergence of the sequence of penalised paths to the reflected solution is obtained.

We finally use the penalisation method to prove that the law at time $t > 0$ of some reflected Gaussian RDE is absolutely continuous with respect to the Lebesgue measure.

Keywords: Gaussian noise; penalisation; reflected rough differential equation; Skorokhod problem

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Sign tests for weak principal directions

DAVY PAINDAVEINE[1,2], JULIEN REMY[1] and THOMAS VERDEBOUT[1]

[1]*ECARES and Département de Mathématique, Université libre de Bruxelles, Avenue F.D. Roosevelt, 50, ECARES, CP114/04, B-1050, Brussels, Belgium.*
E-mail: tverdebo@ulb.ac.be
[2]*Toulouse School of Economics, Université Toulouse Capitole, Esplanade de l'Université, 31080 Toulouse Cedex 06, France*

We consider inference on the first principal direction of a p-variate elliptical distribution. We do so in challenging double asymptotic scenarios for which this direction eventually fails to be identifiable. In order to achieve robustness not only with respect to such weak identifiability but also with respect to heavy tails, we focus on sign-based statistical procedures, that is, on procedures that involve the observations only through their direction from the center of the distribution. We actually consider the generic problem of testing the null hypothesis that the first principal direction coincides with a given direction of $\mathbb{R}^p$. We first focus on weak identifiability setups involving single spikes (that is, involving spectra for which the smallest eigenvalue has multiplicity $p-1$). We show that, irrespective of the degree of weak identifiability, such setups offer local alternatives for which the corresponding sequence of statistical experiments converges in the Le Cam sense. Interestingly, the limiting experiments depend on the degree of weak identifiability. We exploit this convergence result to build optimal sign tests for the problem considered. In classical asymptotic scenarios where the spectrum is fixed, these tests are shown to be asymptotically equivalent to the sign-based likelihood ratio tests available in the literature. Unlike the latter, however, the proposed sign tests are robust to arbitrarily weak identifiability. We show that our tests meet the asymptotic level constraint irrespective of the structure of the spectrum, hence also in possibly multi-spike setups. We fully characterize the non-null asymptotic distributions of the corresponding test statistics under weak identifiability, which allows us to quantify the corresponding local asymptotic powers. Finally, Monte Carlo exercises are conducted to assess the finite-sample relevance of our asymptotic results and a real-data illustration is provided.

Keywords: Le Cam's asymptotic theory of statistical experiments; local asymptotic normality; principal component analysis; sign tests; weak identifiability

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A k-points-based distance for robust geometric inference

CLAIRE BRÉCHETEAU[1,2] and CLÉMENT LEVRARD[3]

[1]*École Centrale de Nantes, 1 Rue de la Noë, 44300 Nantes, France.*
E-mail: claire.brecheteau@ec-nantes.fr; url: http://www.math.sciences.univ-nantes.fr/~brecheteau/
[2]*Laboratoire de Mathématiques Jean Leray, Faculté des Sciences et des Techniques, 2 Chemin de la Houssinière, 44322 Nantes, France*
[3]*Laboratoire de Probabilités, Statistiques et Modélisation, Bâtiment Sophie Germain, Université Paris-Diderot, 75013 Paris, France.*
E-mail: clement.levrard@lpsm.paris; url: http://www.normalesup.org/~levrard

Analyzing the sub-level sets of the distance to a compact submanifold of $\mathbb{R}^d$ is a common method in topological data analysis, to understand its topology. Therefore, topological inference procedures usually rely on a distance estimate based on n sample points (*Discrete Comput. Geom.* **33** (2005) 249–274). In the case where sample points are corrupted by noise, the distance-to-measure function (DTM, *Found. Comput. Math.* **11** (2011) 733–751) is a surrogate for the distance-to-compact-set function. In practice, approximating the homology of its sub-level sets requires to compute the homology of unions of n balls (*Discrete Comput. Geom.* **49** (2013) 22–45; In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms* (2015) 168–180 SIAM), that might become intractable whenever n is large. To simultaneously face the two problems of a large number of points and noise, we introduce the k-power-distance-to-measure function (k-PDTM). This new surrogate for the distance-to-compact is a k-points-based approximation of the DTM. These k points are minimizers of a robustified version of the classical k-means criterion (In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66)* (1967) 281–297 Univ. California Press). The sublevel sets of the k-PDTM consist in unions of k balls, and this distance is also proved robust to noise. We assess the quality of this approximation for k possibly drastically smaller than n, and provide an algorithm to compute this k-PDTM from a sample. Numerical experiments illustrate the good behavior of this k-points approximation in a noisy topological inference framework.

Keywords: minimax rates; quantization; robust distance estimation; topological inference

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Estimation of Monge matrices

JAN-CHRISTIAN HÜTTER[1], CHENG MAO[2], PHILIPPE RIGOLLET[3] and ELINA ROBEVA[4]

[1]*Broad Institute, 415 Main Street, Cambridge, MA, 02142, USA.*
E-mail: jhuetter@broadinstitute.org
[2]*School of Mathematics, Georgia Institute of Technology, Suite 117, 686 Cherry Street, Atlanta, GA, 30332-0160, USA.*
E-mail: cheng.mao@math.gatech.edu
[3]*Department of Mathematics, Massachusetts Institute of Technology, Building 2, Room 106, 77 Massachusetts Avenue, Cambridge, MA, 02139-4307, USA.*
E-mail: rigollet@math.mit.edu
[4]*Department of Mathematics, University of British Columbia, Room 121, 1984 Mathematics Road, Vancouver, BC, V6T 1Z2, Canada.*
E-mail: erobeva@math.ubc.ca

Monge matrices and their permuted versions known as pre-Monge matrices naturally appear in many domains across science and engineering. While the rich structural properties of such matrices have long been leveraged for algorithmic purposes, little is known about their impact on statistical estimation. In this work, we propose to view this structure as a shape constraint and study the problem of estimating a Monge matrix subject to additive random noise. More specifically, we establish the minimax rates of estimation of Monge and pre-Monge matrices. In the case of pre-Monge matrices, the minimax-optimal least-squares estimator is not efficiently computable, and we propose two efficient estimators and establish their rates of convergence. Our theoretical findings are supported by numerical experiments.

Keywords: constrained least-squares estimation; Monge matrices; permuted matrix estimation; shape-constrained estimation

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Nodal lengths in shrinking domains for random eigenfunctions on S^2

ANNA PAOLA TODINO

Ruhr-Universität Bochum, Universitätsstraße 150, 44801 Bochum, Germany.
E-mail: Anna.Todino@ruhr-uni-bochum.de

We investigate the asymptotic behavior of the nodal lines for random spherical harmonics restricted to shrinking domains, in the 2-dimensional case: for example, the length of the zero set $\mathcal{Z}_{\ell,r_\ell} := \mathcal{Z}^{B_{r_\ell}}(T_\ell) = \text{len}(\{x \in S^2 \cap B_{r_\ell} : T_\ell(x) = 0\})$, where B_{r_ℓ} is the spherical cap of radius r_ℓ. We show that the variance of the nodal length is logarithmic in the high energy limit; moreover, it is asymptotically fully equivalent, in the L^2-sense, to the "local sample trispectrum", namely, the integral on the ball of the fourth-order Hermite polynomial. This result extends and generalizes some recent findings for the full spherical case. As a consequence a Central Limit Theorem is established.

Keywords: Berry's cancellation; limit theorem; random eigenfunctions; sample trispectrum

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Area anomaly in the rough path Brownian scaling limit of hidden Markov walks

OLGA LOPUSANSCHI* and DAMIEN SIMON†

Laboratoire de Probabilités, Statistique et Modélisation (UMR 8001), Sorbonne Université and CNRS, 4 place Jussieu, 75005 Paris, France.
E-mail: *olga.lopusanschi@gmail.com; †damien.simon@sorbonne-universite.fr

We study the convergence in rough path topology of a certain class of discrete processes, the hidden Markov walks, to a Brownian motion with an area anomaly. This area anomaly, which is a new object, keeps track of the time-correlation of the discrete models and brings into light the question of embeddings of discrete processes into continuous time. We also identify an underlying combinatorial structure in the hidden Markov walks, which turns out to be a generalization of the occupation time from the classical ergodic theorem in the spirit of rough paths.

Keywords: area anomaly; hidden Markov chains; rough paths

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Concentration inequalities for random tensors

ROMAN VERSHYNIN

Department of Mathematics, University of California, Irvine, Irvine, CA 92617, U.S.A.
E-mail: rvershyn@uci.edu; url: https://www.math.uci.edu/~rvershyn/index.html

We show how to extend several basic concentration inequalities for simple random tensors $X = x_1 \otimes \cdots \otimes x_d$ where all x_k are independent random vectors in $\mathbb{R}^n$ with independent coefficients. The new results have optimal dependence on the dimension n and the degree d. As an application, we show that random tensors are well conditioned: $(1 - o(1))n^d$ independent copies of the simple random tensor $X \in \mathbb{R}^{n^d}$ are far from being linearly dependent with high probability. We prove this fact for any degree $d = o(\sqrt{n/\log n})$ and conjecture that it is true for any $d = O(n)$.

Keywords: concentration inequalities; condition numbers; polynomials; random tensors

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Goodness-of-fit testing for copulas: A distribution-free approach

SAMI UMUT CAN[1,*], JOHN H.J. EINMAHL[2] and ROGER J.A. LAEVEN[1,†]

[1]*Department of Quantitative Economics, University of Amsterdam, P.O. Box 15867, 1001 NJ Amsterdam, The Netherlands.*
E-mail: [*]*s.u.can@uva.nl;* [†]*r.j.a.laeven@uva.nl*
[2]*Department of Econometrics & OR and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.*
E-mail: j.h.j.einmahl@tilburguniversity.edu

Consider a random sample from a continuous multivariate distribution function F with copula C. In order to test the null hypothesis that C belongs to a certain parametric family, we construct an empirical process on the unit hypercube that converges weakly to a standard Wiener process under the null hypothesis. This process can therefore serve as a 'tests generator' for asymptotically distribution-free goodness-of-fit testing of copula families. We also prove maximal sensitivity of this process to contiguous alternatives. Finally, we demonstrate through a Monte Carlo simulation study that our approach has excellent finite-sample performance, and we illustrate its applicability with a data analysis.

Keywords: Copula; distribution-free; goodness-of-fit; Monte Carlo simulation; semi-parametric estimation

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Lévy processes: Concentration function and heat kernel bounds

TOMASZ GRZYWNY[*] and KAROL SZCZYPKOWSKI[†]

Wydział Matematyki, Politechnika Wrocławska, Wyb. Wyspiańskiego 27, 50-370 Wrocław, Poland.
E-mail: [*]*tomasz.grzywny@pwr.edu.pl;* [†]*karol.szczypkowski@pwr.edu.pl*

We investigate densities of vaguely continuous convolution semigroups of probability measures on the Euclidean space. We expose that many typical conditions on the characteristic exponent repeatedly used in the literature of the subject are equivalent to the behaviour of the maximum of the density as a function of time variable. We also prove qualitative lower estimates under mild assumptions on the corresponding jump measure and the characteristic exponent.

Keywords: heat kernel estimates; Lévy process; non-local operator; non-symmetric Markov process; non-symmetric operator; semigroups of measures; transition density

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Some properties of a Cauchy family on the sphere derived from the Möbius transformations

SHOGO KATO[1] and PETER MCCULLAGH[2]

[1]*Institute of Statistical Mathematics, 10-3 Midori-Cho, Tachikawa, Tokyo 190-8562, Japan.*
E-mail: skato@ism.ac.jp
[2]*Department of Statistics, University of Chicago, George Herbert Jones Laboratory, 5747 South Ellis Avenue, Chicago, IL 60637, USA.*
E-mail: pmcc@galton.uchicago.edu

We present some properties of a Cauchy family of distributions on the sphere, which is a spherical extension of the wrapped Cauchy family on the circle. The spherical Cauchy family is closed under the Möbius transformations on the sphere and the parameter of the transformed family is expressed using extended Möbius transformations on the compactified Euclidean space. Stereographic projection transforms the spherical Cauchy family into a multivariate t-family with a certain degree of freedom on Euclidean space. The Möbius transformations and stereographic projection enable us to obtain some results related to the spherical Cauchy family such as an efficient algorithm for random variate generation, a simple form of pivotal statistic and straightforward calculation of probabilities of a region. A method of moments estimator and an asymptotically efficient estimator are expressed in closed form. Maximum likelihood estimation is also straightforward.

Keywords: directional statistics; high dimensional data; stereographic projection; von Mises–Fisher distribution; wrapped Cauchy distribution

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Coupling and perturbation techniques for categorical time series

LIONEL TRUQUET

UMR 9194 CNRS CREST, ENSAI, Campus de Ker-Lann, Rue Blaise Pascal – BP 37203, 35712 BRUZ cedex, France.
E-mail: lionel.truquet@ensai.fr

We present a general approach for studying autoregressive categorical time series models with dependence of infinite order and defined conditional on an exogenous covariate process. To this end, we adapt a coupling approach, developed in the literature for bounding the relaxation speed of a chain with complete connections and from which we derive a perturbation result for non-homogenous versions of such chains. We then study stationarity, ergodicity and dependence properties of some chains with complete connections and exogenous covariates. As a consequence, we obtain a general framework for studying some observation-driven time series models used both in statistics and econometrics but without theoretical support.

Keywords: categorical times series; chains with complete connections; coupling; dependence properties

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Bump detection in the presence of dependency: Does it ease or does it load?

FARIDA ENIKEEVA[1,2], AXEL MUNK[3,4,5,*], MARKUS POHLMANN[3,†] and FRANK WERNER[6]

[1]*Laboratoire de Mathématiques et Applications, CNRS UMR 7348, Université de Poitiers, Site du Futuroscope – Téléport 2, 11 Boulevard Marie et Pierre Curie, Bâtiment H3 – TSA 61125, 86073 Poitiers Cedex 9, France.*
E-mail: farida.enikeeva@math.univ-poitiers.fr
[2]*The A.A. Kharkevich Institute for Information Transmission Problems, Russian Academy of Science, Bolshoy Karentny per. 19-1, 127051 Moscow, Russia*
[3]*Institute for Mathematical Stochastics, University of Goettingen, Goldschmidtstraße 7, 37077 Göttingen, Germany.*
*E-mail: *munk@math.uni-goettingen.de; †markus.pohlmann@mathematik.uni-goettingen.de*
[4]*Felix Bernstein Institute for Mathematical Statistics in the Biosciences, University of Goettingen, Goldschmidtstraße 7, 37077 Göttingen, Germany*
[5]*Statistical Inverse Problems in Biophysics Group, Max Planck Institute for Biophysical Chemistry, Am Faßberg 11, 37077 Göttingen, Germany*
[6]*Institut für Mathematik, University of Wuerzburg, Emil-Fischer-Str. 30, 97074 Würzburg, Germany.*
E-mail: frank.werner@mathematik.uni-wuerzburg.de

We provide the asymptotic minimax detection boundary for a bump, i.e. an abrupt change, in the mean function of a stationary Gaussian process. This will be characterized in terms of the asymptotic behavior of the bump length and height as well as the dependency structure of the process. A major finding is that the asymptotic minimax detection boundary is generically determined by the value of its spectral density at zero. Finally, our asymptotic analysis is complemented by non-asymptotic results for AR(p) processes and confirmed to serve as a good proxy for finite sample scenarios in a simulation study. Our proofs are based on laws of large numbers for non-independent and non-identically distributed arrays of random variables and the asymptotically sharp analysis of the precision matrix of the process.

Keywords: ARMA processes; change point detection; minimax testing; time series; Toeplitz matrices; weak laws of large numbers

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