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Aims and Scope

Issued four times per year, BERNOULLI is the flagship journal of the Bernoulli Society for Mathematical Statistics and Probability. The journal aims at publishing original research contributions of the highest quality in all subfields of Mathematical Statistics and Probability. The main emphasis of Bernoulli is on theoretical work, yet discussion of interesting applications in relation to the proposed methodology is also welcome.

Bernoulli Society for Mathematical Statistics and Probability

The Bernoulli Society was founded in 1973. It is an autonomous Association of the International Statistical Institute, ISI. According to its statutes, the object of the Bernoulli Society is the advancement, through international contacts, of the sciences of probability (including the theory of stochastic processes) and mathematical statistics and of their applications to all those aspects of human endeavour which are directed towards the increase of natural knowledge and the welfare of mankind.

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Small sample spaces for Gaussian processes

TONI KARVONEN[1,2,a]

[1]*The Alan Turing Institute, 96 Euston Road, London NW1 2DB, United Kingdom*
[2]*University of Helsinki, Department of Mathematics and Statistics, Pietari Kalmin katu 5, 00560 Helsinki, Finland,* [a]*toni.karvonen@helsinki.fi*

It is known that the membership in a given reproducing kernel Hilbert space (RKHS) of the samples of a Gaussian process X is controlled by a certain nuclear dominance condition. However, it is less clear how to identify a "small" set of functions (not necessarily a vector space) that contains the samples. This article presents a general approach for identifying such sets. We use *scaled RKHSs*, which can be viewed as a generalisation of Hilbert scales, to define the *sample support set* as the largest set which is contained in every element of full measure under the law of X in the σ-algebra induced by the collection of scaled RKHS. This potentially non-measurable set is then shown to consist of those functions that can be expanded in terms of an orthonormal basis of the RKHS of the covariance kernel of X and have their squared basis coefficients bounded away from zero and infinity, a result suggested by the Karhunen–Loève theorem.

Keywords: Gaussian processes; sample path properties; reproducing kernel Hilbert spaces

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Bayes-optimal prediction with frequentist coverage control

PETER HOFF[a]

Statistical Science, Duke University, Durham, USA, [a]*peter.hoff@duke.edu*

This article illustrates how indirect or prior information can be optimally used to construct a prediction region that maintains a target frequentist coverage rate. If the indirect information is accurate, the volume of the prediction region is lower on average than that of other regions with the same coverage rate. Even if the indirect information is inaccurate, the resulting region still maintains the target coverage rate. Such a prediction region can be constructed for models that have a complete sufficient statistic, which includes many widely-used parametric and nonparametric models. Particular examples include a Bayes-optimal conformal prediction procedure that maintains a constant coverage rate across distributions in a nonparametric model, as well as a prediction procedure for the normal linear regression model that can utilize a regularizing prior distribution, yet maintain a frequentist coverage rate that is constant as a function of the model parameters and explanatory variables. No results in this article rely on asymptotic approximations.

Keywords: Conformal prediction; disintegration; hypothesis testing; Neyman-Pearson lemma; tolerance region

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Concentration inequality for U-statistics of order two for uniformly ergodic Markov chains

QUENTIN DUCHEMIN[1,a], YOHANN DE CASTRO[2,c] and CLAIRE LACOUR[1,b]

[1]*LAMA, Univ Gustave Eiffel, CNRS, Marne-la-Vallée, France,* [a]*quentin.duchemin@univ-eiffel.fr,* [b]*claire.lacour@univ-eiffel.fr*
[2]*Institut Camille Jordan, École Centrale de Lyon, Lyon, France,* [c]*yohann.de-castro@ec-lyon.fr*

We prove a new concentration inequality for U-statistics of order two for uniformly ergodic Markov chains. Working with bounded and π-*canonical* kernels, we show that we can recover the convergence rate of Arcones and Giné who proved a concentration result for U-statistics of independent random variables and canonical kernels. Our result allows for a dependence of the kernels $h_{i,j}$ with the indexes in the sums, which prevents the use of standard blocking tools. Our proof relies on an inductive analysis where we use martingale techniques, uniform ergodicity, Nummelin splitting and Bernstein's type inequality.

Assuming further that the Markov chain starts from its invariant distribution, we prove a Bernstein-type concentration inequality that provides sharper convergence rate for small variance terms.

Keywords: U-statistics; Markov chains; concentration inequality

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Determinantal point processes in the flat limit

SIMON BARTHELMÉ[1,a], NICOLAS TREMBLAY[1], KONSTANTIN USEVICH[2] and PIERRE-OLIVIER AMBLARD[1]

[1]*CNRS, Univ. Grenoble Alpes, Grenoble INP, GIPSA-lab, Grenoble, France,* [a]*simon.barthelme@gipsa-lab.fr*
[2]*Université de Lorraine and CNRS, CRAN (Centre de Recherche en Automatique en Nancy), Grenoble, France*

Determinantal point processes (DPPs) are repulsive point processes where the interaction between points depends on the determinant of a positive-semi definite matrix.

In this paper, we study the limiting process of L-ensembles based on kernel matrices, when the kernel function becomes flat (so that every point interacts with every other point, in a sense). We show that these limiting processes are best described in the formalism of extended L-ensembles and partial projection DPPs, and the exact limit depends mostly on the smoothness of the kernel function. In some cases, the limiting process is even universal, meaning that it does not depend on specifics of the kernel function, but only on its degree of smoothness.

Since flat-limit DPPs are still repulsive processes, this implies that practically useful families of DPPs exist that do not require a spatial length-scale parameter.

Keywords: Determinantal point processes; kernel methods; flat limit

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Central limit theorem of linear spectral statistics of high-dimensional sample correlation matrices

YANQING YIN[1,a], SHURONG ZHENG[2,b] and TINGTING ZOU[3,c]

[1]*Chongqing University, Chongqing, China,* [a]*yinyq799@nenu.edu.cn*
[2]*Northeast Normal University, Changchun, China,* [b]*zhengsr@nenu.edu.cn*
[3]*Jilin University, Changchun, China,* [c]*zoutt260@nenu.edu.cn*

A high-dimensional sample correlation matrix is an important random matrix in multivariate statistical analysis. Its central limit theory is one of the main theoretical bases for making statistical inferences on high-dimensional correlation matrices. Under the high-dimensional framework in which the data dimension tends to infinity proportionally with the sample size, we establish the central limit theorems (CLT) for the linear spectral statistics (LSS) of sample correlation matrices in two settings: (1) the population follows an independent component structure; (2) the population follows an elliptical structure, including some heavy-tailed distributions. The results show that the CLTs of the LSS of the sample correlation matrices are very different in the two settings. In particular, even if the population correlation matrix is an identity matrix, the CLTs are different in the two settings. An application of our two established CLTs is provided.

Keywords: Sample correlation matrix; high-dimensional; independent component structure; elliptical distribution; central limit theorem; random matrix theory

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Large deviations of extremes in branching random walk with regularly varying displacements

AYAN BHATTACHARYA[a]

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai, India, [a]*ayanbh@math.iitb.ac.in*

In this article, we consider a branching random walk on the real-line where displacements coming from the same parent have jointly regularly varying tails. The genealogical structure is assumed to be a supercritical Galton-Watson tree, satisfying Kesten-Stigum condition. We study the large deviations of the extremal process, formed by the appropriately normalized positions in the n-th generation and show that the large extreme-positions form clusters in the limit. As a consequence of this, we also study the large deviations of the maximum among positions at the n-th generation.

Keywords: Branching random walk; extreme values; regular variation; point process; maximum position; large deviations

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Exponential ergodicity for non-dissipative McKean-Vlasov SDEs

FENG-YU WANG[1,2,a]

[1]*Center for Applied Mathematics, Tianjin University, Tianjin 300072, China.*
[2]*Department of Mathematics, Swansea University, Bay Campus, Swansea, SA1 8EN, United Kingdom,*
[a]*wangfy@tju.edu.cn*

Under Lyapunov and monotone conditions, the exponential ergodicity in the induced Wasserstein quasi-distance is proved for a class of non-dissipative McKean-Vlasov SDEs, which strengthen some recent results established under dissipative conditions in long distance. Moreover, when the SDE is order-preserving, the exponential ergodicity is derived in the Wasserstein distance induced by one-dimensional increasing functions chosen according to the coefficients of the equation.

Keywords: Exponential ergodicity; McKean-Vlasov SDEs; Lyapunov condition; coupling method

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Small deviation estimates for the largest eigenvalue of Wigner matrices

LÁSZLÓ ERDŐS[a] and YUANYUAN XU[b]

Am Campus 1, IST Austria, Klosterneuburg, Austria, [a]*lerdos@ist.ac.at,* [b]*yuanyuan.xu@ist.ac.at*

We establish precise right-tail small deviation estimates for the largest eigenvalue of real symmetric and complex Hermitian matrices whose entries are independent random variables with uniformly bounded moments. The proof relies on a Green function comparison along a continuous interpolating matrix flow for a long time. Less precise estimates are also obtained in the left tail.

Keywords: Largest eigenvalue; Tracy-Widom law; small deviation; Wigner matrix

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Gibbs posterior concentration rates under sub-exponential type losses

NICHOLAS SYRING[1,a] and RYAN MARTIN[2,b]

[1]*Department of Statistics, Iowa State University, Ames, IA USA,* [a]*nsyring@iastate.edu*
[2]*Department of Statistics, North Carolina State University, Raleigh, NC USA,* [b]*rgmarti3@ncsu.edu*

Bayesian posterior distributions are widely used for inference, but their dependence on a statistical model creates some challenges. In particular, there may be lots of nuisance parameters that require prior distributions and posterior computations, plus a potentially serious risk of model misspecification bias. Gibbs posterior distributions, on the other hand, offer direct, principled, probabilistic inference on quantities of interest through a loss function, not a model-based likelihood. Here we provide simple sufficient conditions for establishing Gibbs posterior concentration rates when the loss function is of a sub-exponential type. We apply these general results in a range of practically relevant examples, including mean regression, quantile regression, and sparse high-dimensional classification. We also apply these techniques in an important problem in medical statistics, namely, estimation of a personalized minimum clinically important difference.

Keywords: Classification; generalized Bayes; high-dimensional problem; M-estimation; model misspecification

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Randomized empirical processes by algebraic groups, and tests for weak null hypotheses

DENNIS DOBLER[a]

Department of Mathematics, Faculty of Science, Vrije Universiteit Amsterdam, The Netherlands, [a]*d.dobler@vu.nl*

Randomization tests are based on a re-randomization of existing data to gain data-dependent critical values that lead to exact hypothesis tests under special circumstances. However, it is not always possible to re-randomize data in accordance with the physical randomization from which the data have been obtained. As a consequence, statistical tests typically cannot control the type I error probability if the available sample is not overly large. Still, similarly as the bootstrap, data re-randomization can be used to improve the type I error control. However, in contrast to bootstrap and permutation test theory, a general asymptotic theory under weak null hypotheses has not yet been developed for randomization tests. It is thus the aim of this paper to provide a conveniently applicable theory on the asymptotic validity of randomization tests with test statistics that are asymptotically normal under weak null hypotheses. This can also be used to justify the asymptotic reliability of randomization-based confidence intervals.

To achieve this, the present article creates a link between two well-established fields in mathematical statistics: empirical processes and inference based on randomization via algebraic groups. To this end, a broadly applicable conditional weak convergence theorem is developed for empirical processes that are based on randomized observations. Random elements of an algebraic group are applied to the data vectors from which the randomized version of a statistic is derived. Combining a variant of the functional delta-method with a suitable studentization of the statistic, asymptotically exact hypothesis tests can be deduced, while the finite sample exactness property under group-invariant sub-hypotheses is preserved. The methodology is exemplified with three examples: the Pearson correlation coefficient, a Mann-Whitney effect based on right-censored paired data, and a competing risks analysis. The practical usefulness of the approaches is assessed through simulation studies and an application to data from patients suffering from diabetic retinopathy.

Keywords: Weak convergence; empirical process; exact testing; functional delta-method; randomization inference

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Two-timescale stochastic gradient descent in continuous time with applications to joint online parameter estimation and optimal sensor placement

LOUIS SHARROCK[a] and NIKOLAS KANTAS[b]

Department of Mathematics, Imperial College London, South Kensington, London, SW7 2AZ, United Kingdom,
[a]*louis.sharrock16@imperial.ac.uk,* [b]*n.kantas@imperial.ac.uk*

In this paper, we establish the almost sure convergence of two-timescale stochastic gradient descent algorithms in continuous time under general noise and stability conditions, extending well known results in discrete time. We analyse algorithms with additive noise and those with non-additive noise. In the non-additive case, our analysis is carried out under the assumption that the noise is a continuous-time Markov process, controlled by the algorithm states. The algorithms we consider can be applied to a broad class of bilevel optimisation problems. We study one such problem in detail, namely, the problem of joint online parameter estimation and optimal sensor placement for a partially observed diffusion process. We demonstrate how this can be formulated as a bilevel optimisation problem, and propose a solution in the form of a continuous-time, two-timescale, stochastic gradient descent algorithm. Furthermore, under suitable conditions on the latent signal, the filter, and the filter derivatives, we establish almost sure convergence of the online parameter estimates and optimal sensor placements to the stationary points of the asymptotic log-likelihood and asymptotic filter covariance, respectively. We also provide numerical examples, illustrating the application of the proposed methodology to a partially observed Beneš equation, and a partially observed stochastic advection-diffusion equation.

Keywords: Two-timescale stochastic approximation; stochastic gradient descent; recursive maximum likelihood; online parameter estimation; optimal sensor placement; Beneš filter; Kalman-Bucy filter

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Fisher's measure of variability in repeated samples

POLY H. DA SILVA[1,a], ARASH JAMSHIDPEY[2,b], PETER MCCULLAGH[3,c] and SIMON TAVARÉ[1,4,d]

[1]*Department of Statistics, Columbia University, 1255 Amsterdam Ave, New York, NY 10027, USA,* [a]*phd2120@columbia.edu*
[2]*Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, USA,* [b]*aj2963@columbia.edu*
[3]*Department of Statistics, University of Chicago, 5747 S Ellis Ave, Chicago, IL 60637, USA,* [c]*pmcc@galton.uchicago.edu*
[4]*Irving Institute for Cancer Dynamics, Columbia University, Schermerhorn Hall, Suite 601, 1190 Amsterdam Avenue, New York, NY 10027, USA,* [d]*st3193@columbia.edu*

Fisher (1943) claimed that the expected value of the sample variance of the number of species found in large samples, each of n specimens taken from the same population, is asymptotically $\theta \log 2$. This is at odds with the value $\theta \log n$ obtained directly from the Ewens Sampling Formula (ESF), where θ specifies the rate at which new species are found. To resolve this apparent contradiction, we assume the species frequency spectrum in the population is determined by the ESF and that the samples are disjoint subsets drawn sequentially from this single population. We find an explicit formula for the required expected value for p samples of arbitrary size; in the limit of large equally-sized samples, it indeed has the value $\theta \log 2$. We obtain limit theorems for the sample variance of p samples of size n under various limiting regimes as p, n or both tend to ∞. We discuss further the behavior of the number of species present in all samples, and revisit Fisher's log-series distribution as the limiting distribution of the number of specimens observed in typical species in a future, large sample.

Keywords: Ewens Sampling Formula; log-series model; sequential sampling; exchangeability; Chinese Restaurant Process; Poisson approximation

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Optimal Bayesian estimation of Gaussian mixtures with growing number of components

ILSANG OHN[1,a] and LIZHEN LIN[2,b]

[1]*Department of Statistics, Inha University, Incheon, Republic of Korea,* [a]*ilsang.ohn@inha.ac.kr*
[2]*Department of Applied and Computational Mathematics and Statistics, The University of Notre Dame, Indiana, USA,* [b]*lizhen.lin@nd.edu*

We study Bayesian estimation of finite mixture models in a general setup where the number of components is unknown and allowed to grow with the sample size. An assumption on growing number of components is a natural one as the degree of heterogeneity present in the sample can grow and new components can arise as sample size increases, allowing full flexibility in modeling the complexity of data. This however will lead to a high-dimensional model which poses great challenges for estimation. We novelly employ the idea of a sample size dependent prior in a Bayesian model and establish a number of important theoretical results. We first show that under mild conditions on the prior, the posterior distribution concentrates around the true mixing distribution at a near optimal rate with respect to the Wasserstein distance. Under a separation condition on the true mixing distribution, we further show that a better and adaptive convergence rate can be achieved, and the number of components can be consistently estimated. Furthermore, we derive optimal convergence rates for the higher-order mixture models where the number of components diverges arbitrarily fast. In addition, we suggest a simple recipe for using Dirichlet process (DP) mixture prior for estimating the finite mixture models and provide theoretical guarantees. In particular, we provide a novel solution for adopting the number of clusters in a DP mixture model as an estimate of the number of components in a finite mixture model. Simulation study and real data applications are carried out demonstrating the utilities of our method.

Keywords: Gaussian mixtures; finite mixture models; growing number of components; mixing distribution estimation; posterior contraction rates; Dirichlet processes

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A Berry-Esseen bound with (almost) sharp dependence conditions

MORITZ JIRAK[a]

University of Vienna, [a]*moritz.jirak@univie.ac.at*

Suppose that the (normalised) partial sum of a stationary sequence converges to a standard normal random variable. Given sufficiently moments, when do we have a rate of convergence of $n^{-1/2}$ in the uniform metric, in other words, when do we have the optimal Berry-Esseen bound? We study this question in a quite general framework and find the (almost) sharp dependence conditions. The result applies to many different processes and dynamical systems. As specific, prominent examples, we study functions of the doubling map 2x mod 1, the left random walk on the general linear group and functions of linear processes.

Keywords: Berry-Esseen; weak dependence

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Combinatorial Bernoulli factories

RAD NIAZADEH[1,a], RENATO PAES LEME[2,b] and JON SCHNEIDER[2,c]

[1] *The University of Chicago Booth School of Business, Chicago, USA,* [a] *rad.niazadeh@chicagobooth.edu*
[2] *Google Research, New York City, USA,* [b] *renatoppl@google.com,* [c] *jschneider@google.com*

A Bernoulli factory is an algorithmic procedure for exact sampling of certain random variables having only Bernoulli access to their parameters. Bernoulli access to a parameter $p \in [0,1]$ means the algorithm does not know p, but has sample access to independent draws of a Bernoulli random variable with mean equal to p. In this paper, we study the problem of Bernoulli factories for polytopes: given Bernoulli access to a vector $x \in \mathcal{P}$ for a given polytope $\mathcal{P} \subset [0,1]^n$, output a randomized vertex such that the expected value of the i-th coordinate is *exactly* equal to x_i. For example, for the special case of the perfect matching polytope, one is given Bernoulli access to the entries of a doubly stochastic matrix $[x_{ij}]$ and asked to sample a matching such that the probability of each edge (i,j) be present in the matching is exactly equal to x_{ij}.

We show that a polytope $\mathcal{P}$ admits a Bernoulli factory if and and only if $\mathcal{P}$ is the intersection of $[0,1]^n$ with an affine subspace. Our construction is based on an algebraic formulation of the problem, involving identifying a family of Bernstein polynomials (one per vertex) that satisfy a certain algebraic identity on $\mathcal{P}$. The main technical tool behind our construction is a connection between these polynomials and the geometry of zonotope tilings.

We apply these results to construct an explicit factory for the perfect matching polytope. The resulting factory is deeply connected to the combinatorial enumeration of arborescences and may be of independent interest. For the k-uniform matroid polytope, we recover a sampling procedure known in statistics as Sampford sampling.*

Keywords: Bernoulli factories; exact simulation; combinatorial polytopes; Sampford sampling

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On the eigenstructure of covariance matrices with divergent spikes

SIMONA DIACONU[a]

Department of Mathematics, Stanford University, Stanford, USA, [a]*sdiaconu@stanford.edu*

For a generalization of Johnstone's spiked model, a covariance matrix with eigenvalues all one but M of them, the number of features N comparable to the number of samples $n : N = N(n), M = M(n), \gamma^{-1} \leq \frac{N}{n} \leq \gamma$ where $\gamma \in (0, \infty)$, we obtain consistency rates in the form of CLTs for separated spikes tending to infinity fast enough whenever M grows slightly slower than $n : \lim_{n\to\infty} \frac{\sqrt{\log n}}{\log \frac{n}{M(n)}} = 0$. Our results fill a gap in the existing literature in which the largest range covered for the number of spikes has been $o(n^{1/6})$ and reveal a certain degree of flexibility for the centering in these CLTs inasmuch as it can be empirical, deterministic, or a sum of both. Furthermore, we derive consistency rates of their corresponding empirical eigenvectors to their true counterparts, which turn out to depend on the relative growth of these eigenvalues.

Keywords: Covariance matrices; divergent spikes; eigenvalue CLTs

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Stability of the Poincaré constant

SERRES JORDAN[a]

Institut de Mathématiques de Toulouse, [a]*jordan.serres@math.univ-toulouse.fr*

We study stability of the sharp Poincaré constant of the invariant probability measure of a reversible diffusion process satisfying some natural conditions. The proof is based on the spectral interpretation of Poincaré inequalities and Stein's method. In particular, these results are applied to gamma distributions, to the Brownian motion on spheres and to the Brascamp-Lieb inequality for one-dimensional log-concave measures.

Keywords: Poincaré inequalities; Stein's method; spectral gap

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Kernel based Dirichlet sequences

PATRIZIA BERTI[1,a], EMANUELA DREASSI[2,b], FABRIZIO LEISEN[3,c], LUCA PRATELLI[4,d] and PIETRO RIGO[5,e]

[1]*Dipartimento di Matematica Pura ed Applicata "G. Vitali", Università di Modena e Reggio-Emilia, via Campi 213/B, 41100 Modena, Italy,* [a]*patrizia.berti@unimore.it*
[2]*Dipartimento di Statistica, Informatica, Applicazioni, Università di Firenze, viale Morgagni 59, 50134 Firenze, Italy,* [b]*emanuela.dreassi@unifi.it*
[3]*School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, UK,* [c]*fabrizio.leisen@gmail.com*
[4]*Accademia Navale, viale Italia 72, 57100 Livorno, Italy,* [d]*pratel@mail.dm.unipi.it*
[5]*Dipartimento di Scienze Statistiche "P. Fortunati", Università di Bologna, via delle Belle Arti 41, 40126 Bologna, Italy,* [e]*pietro.rigo@unibo.it*

Let $X = (X_1, X_2, \ldots)$ be a sequence of random variables with values in a standard space $(S, \mathcal{B})$. Suppose

$$X_1 \sim \nu \quad \text{and} \quad P\big(X_{n+1} \in \cdot \mid X_1, \ldots, X_n\big) = \frac{\theta \nu(\cdot) + \sum_{i=1}^n K(X_i)(\cdot)}{n+\theta} \qquad \text{a.s.}$$

where $\theta > 0$ is a constant, ν a probability measure on $\mathcal{B}$, and K a random probability measure on $\mathcal{B}$. Then, X is exchangeable whenever K is a regular conditional distribution for ν given any sub-σ-field of $\mathcal{B}$. Under this assumption, X enjoys all the main properties of classical Dirichlet sequences, including Sethuraman's representation, conjugacy property, and convergence in total variation of predictive distributions. If μ is the weak limit of the empirical measures, conditions for μ to be a.s. discrete, or a.s. non-atomic, or $\mu \ll \nu$ a.s., are provided. Two CLT's are proved as well. The first deals with stable convergence while the second concerns total variation distance.

Keywords: Bayesian nonparametrics; central limit theorem; Dirichlet sequence; exchangeability; predictive distribution; random probability measure; regular conditional distribution

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Convergence of the one-dimensional contact process with two types of particles and priority

MARIELA PENTÓN MACHADO[a]

Instituto de Matemática e Estatística. Universidade de São Paulo, SP, Brazil, [a]*mariela@dme.ufrj.br*

We consider a symmetric finite-range contact process on $\mathbb{Z}$ with two types of particles (or infections), which propagate according to the same supercritical rate and die (or heal) at rate 1. Particles of type 1 can enter any site in $(-\infty,0]$ that is empty or occupied by a particle of type 2 and, analogously, particles of type 2 can enter any site in $[1,\infty)$ that is empty or occupied by a particle of type 1. Also, at most one particle can occupy each site. We prove that the process with initial configuration $\mathbb{1}_{(-\infty,0]} + 2\mathbb{1}_{[1,\infty)}$ converges in distribution to an invariant measure different from the non trivial invariant measure of the classic contact process. In addition, we prove that for any initial configuration the process converges to a convex combination of four invariant measures.

Keywords: Contact process; percolation

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Detecting approximate replicate components of a high-dimensional random vector with latent structure

XIN BING[1,a], FLORENTINA BUNEA[2,b] and MARTEN WEGKAMP[2,3,c]

[1]*Department of Statistical Sciences, University of Toronto, Toronto, Ontario, Canada,* [a]*xin.bing@utoronto.ca*
[2]*Department of Statistics and Data Science, Cornell University, Ithaca, New York, USA,* [b]*fb238@cornell.edu*
[3]*Department of Mathematics, Cornell University, Ithaca, New York, USA,* [c]*mhw73@cornell.edu*

High-dimensional feature vectors are likely to contain sets of measurements that are approximate replicates of one another. In complex applications, or automated data collection, these feature sets are not known a priori, and need to be determined.

This work proposes a class of latent factor models on the observed, high-dimensional, random vector $X \in \mathbb{R}^p$, for defining, identifying and estimating the index set of its approximately replicate components. The model class is parametrized by a $p \times K$ loading matrix A that contains a hidden sub-matrix whose rows can be partitioned into groups of parallel vectors. Under this model class, a set of approximate replicate components of X corresponds to a set of parallel rows in A: these entries of X are, up to scale and additive error, the same linear combination of the K latent factors; the value of K is itself unknown.

The problem of finding approximate replicates in X reduces to identifying, and estimating, the location of the hidden sub-matrix within A, and of the partition $\mathcal{H}$ of its row index set H. Both H and $\mathcal{H}$ can be fully characterized in terms of a new family of criteria based on the correlation matrix of X, and their identifiability, as well as that of the unknown latent dimension K, are obtained as consequences. The constructive nature of the identifiability arguments enables computationally efficient procedures, with consistency guarantees.

Furthermore, when the loading matrix A has a particular sparse structure, provided by the errors-in-variable parametrization, the difficulty of the problem is elevated. The task becomes that of separating out groups of parallel rows that are proportional to canonical basis vectors from other, possibly dense, parallel rows in A. This is met under a scale assumption, via a principled way of selecting the target row indices, guided by the successive maximization of Schur complements of appropriate covariance matrices. The resulting procedure is an enhanced version of that developed for recovering general parallel rows in A. It is also computationally efficient, consistent. It has immediate applications to latent space overlapping clustering and the estimation of loading matrices that satisfy a canonical parametrization.

Keywords: High-dimensional statistics; identification; latent factor model; matrix factorization; replicate measurements; pure variables; overlapping clustering

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Spectral representations of characteristic functions of discrete probability laws

IVAN ALEXEEV[1,a] and ALEXEY KHARTOV[2,b]

[1]*The Euler International Mathematical Institute, 10 Pesochnaya nab., 197022 St. Petersburg, Russia,* [a]*vanyalexeev@list.ru*
[2]*Smolensk State University, 4 Przhevalsky st., 214000 Smolensk, Russia,* [b]*alexeykhartov@gmail.com*

We consider discrete probability laws on the real line, whose characteristic functions are separated from zero. This class includes arbitrary discrete infinitely divisible laws and lattice probability laws, whose characteristic functions have no zeroes on the real line. We show that characteristic functions of such laws admit spectral Lévy–Khinchine type representation with non-monotonic Lévy spectral function. We also apply the representations of such laws to obtain limit and compactness theorems with convergence in variation to probability laws from this class.

Keywords: Discrete probability laws; characteristic functions; spectral Lévy–Khinchine type representations; quasi-infinitely divisible laws; convergence in variation; relative compactness; stochastic compactness

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Smoothing distributions for conditional Fleming–Viot and Dawson–Watanabe diffusions

FILIPPO ASCOLANI[1,a], ANTONIO LIJOI[1,b] and MATTEO RUGGIERO[2,c]

[1] *Department of Decision Sciences and BIDSA, Bocconi University, Milan, Italy,*
[a] *filippo.ascolani@phd.unibocconi.it,* [b] *antonio.lijoi@unibocconi.it*
[2] *ESOMAS Department and Collegio Carlo Alberto, University of Torino, Torino, Italy,*
[c] *matteo.ruggiero@unito.it*

We study the distribution of the unobserved states of two measure-valued diffusions of Fleming–Viot and Dawson–Watanabe type, conditional on observations from the underlying populations collected at past, present and future times. If seen as nonparametric hidden Markov models, this amounts to finding the smoothing distributions of these processes, which we show can be explicitly described in recursive form as finite mixtures of laws of Dirichlet and gamma random measures respectively. We characterize the time-dependent weights of these mixtures, accounting for potentially different time intervals between data collection times, and fully describe the implications of assuming a discrete or a nonatomic distribution for the underlying process that drives mutations. In particular, we show that with a nonatomic mutation offspring distribution, the inference automatically upweights mixture components that carry, as atoms, observed types shared at different collection times. The predictive distributions for further samples from the population conditional on the data are also identified and shown to be mixtures of generalized Pólya urns, conditionally on a latent variable in the Dawson–Watanabe case.

Keywords: Dirichlet process; duality; gamma random measures; hidden Markov model; optimal filtering; prediction

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Hopf type lemmas for subsolutions of integro-differential equations

TOMASZ KLIMSIAK[1,2,a] and TOMASZ KOMOROWSKI[1,b]

[1]*Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland,* [a]*tomas@mat.umk.pl*
[2]*Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland,* [b]*komorow@hektor.umcs.lublin.pl*

In the paper we prove a lower bound for subsolutions of the integro-differential equation: $-Au + cu = 0$ in a domain D. It states that there exists a Borel function ψ, strictly positive on D, depending only on the coefficients of the operator A, c and D such that for any subsolution $u(\cdot)$, that satisfies $\sup_{y\in D_S} u(y) \ge 0$, one can find a constant $a > 0$ (that in general depends on u), for which $\sup_{y\in D_S} u(y) - u(x) \ge a\psi(x)$, $x \in D$. The bound is valid for a wide class of Lévy type integro-differential operators A, non-negative, bounded and measurable function c and a quite general domain $D \subset \mathbb{R}^d$. Here D_S is a certain set containing the closure of D and determined by the support of the Levy jump measure associated with A. In some cases a non-negative eigenfunction corresponding to the operator in D can be admitted as the function ψ. In particular, this occurs when the transition probability semigroup associated with A is ultracontractive. The main assumptions made about A are: there exists a strong Markov solution to the martingale problem associated with the operator and its resolvent satisfies some minorization condition. This type of a result we call the *generalized Hopf lemma*.

Keywords: Integro-differential elliptic equation; weak subsolution; maximum principle; the Hopf lemma

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Rudin extension theorems on product spaces, turning bands, and random fields on balls cross time

EMILIO PORCU[1,a], SAMUEL F. FENG[1,b], XAVIER EMERY[2,c] and ANA P. PERON[3,d]

[1]*Department of Mathematics & Research And Data Intelligence Support Center R-DISC, Khalifa University of Science and Technology, Abu Dhabi, United Arab Emirates,* [a]*emilio.porcu@ku.ac.ae,* [b]*samuel.feng@ku.ac.ae*
[2]*Department of Mining Engineering & Advanced Mining Technology Center, University of Chile, Santiago, Chile,* [c]*xemery@ing.uchile.cl*
[3]*Department of Mathematics, Institute of Mathematical and Computer Sciences, University of São Paulo, Brazil,* [d]*apperon@icmc.usp.br*

Characteristic functions that are radially symmetric have a dual interpretation, as they can be used as the isotropic correlation functions of spatial random fields. Extensions of isotropic correlation functions from balls into d-dimensional Euclidean spaces, $\mathbb{R}^d$, have been understood after Rudin. Yet, extension theorems on product spaces are elusive, and a counterexample provided by Rudin on rectangles suggests that the problem is challenging. This paper provides extension theorems for multiradial characteristic functions that are defined in balls embedded in $\mathbb{R}^d$ cross, either $\mathbb{R}^{d'}$ or the unit sphere $\mathbb{S}^{d'}$ embedded in $\mathbb{R}^{d'+1}$, for any two positive integers d and d'. We then examine Turning Bands operators that provide bijections between the class of multiradial correlation functions in given product spaces, and multiradial correlations in product spaces having different dimensions. The combination of extension theorems with Turning Bands provides a connection with random fields that are defined in balls cross linear or circular time.

Keywords: Characteristic functions; Rudin's extensions; random fields; turning bands

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Equivalence of measures and asymptotically optimal linear prediction for Gaussian random fields with fractional-order covariance operators

DAVID BOLIN[1,a] and KRISTIN KIRCHNER[2,b]

[1] *CEMSE Division, King Abdullah University of Science and Technology, Thuwal, Saudi Arabia,*
[a] *david.bolin@kaust.edu.sa*
[2] *Delft Institute of Applied Mathematics, Delft University of Technology, Delft, The Netherlands,*
[b] *k.kirchner@tudelft.nl*

We consider two Gaussian measures $\mu, \widetilde{\mu}$ on a separable Hilbert space, with fractional-order covariance operators $A^{-2\beta}$ and $\widetilde{A}^{-2\widetilde{\beta}}$, respectively, and derive necessary and sufficient conditions on $A, \widetilde{A}$ and $\beta, \widetilde{\beta} > 0$ for I. equivalence of the measures μ and $\widetilde{\mu}$, and II. uniform asymptotic optimality of linear predictions for μ based on the misspecified measure $\widetilde{\mu}$. These results hold, e.g., for Gaussian processes on compact metric spaces. As an important special case, we consider the class of generalized Whittle–Matérn Gaussian random fields, where A and $\widetilde{A}$ are elliptic second-order differential operators, formulated on a bounded Euclidean domain $\mathcal{D} \subset \mathbb{R}^d$ and augmented with homogeneous Dirichlet boundary conditions. Our outcomes explain why the predictive performances of stationary and non-stationary models in spatial statistics often are comparable, and provide a crucial first step in deriving consistency results for parameter estimation of generalized Whittle–Matérn fields.

Keywords: Gaussian measures; kriging; elliptic differential operators; Whittle–Matérn fields

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Nonstationary fractionally integrated functional time series

DEGUI LI[1,a], PETER M. ROBINSON[2,b] and HAN LIN SHANG[3,c]

[1]*Department of Mathematics, University of York, York, YO10 5DD, UK,* [a]*degui.li@york.ac.uk*
[2]*Department of Economics, London School of Economics, London WC2A 2AE, UK,* [b]*P.M.Robinson@lse.ac.uk*
[3]*Department of Actuarial Studies and Business Analytics, Macquarie University, Sydney NSW 2109, Australia,* [c]*hanlin.shang@mq.edu.au*

We study a functional version of nonstationary fractionally integrated time series, covering the functional unit root as a special case. The time series taking values in an infinite-dimensional separable Hilbert space are projected onto a finite number of sub-spaces, the level of nonstationarity allowed to vary over them. Under regularity conditions, we derive a weak convergence result for the projection of the fractionally integrated functional process onto the asymptotically dominant sub-space, which retains most of the sample information carried by the original functional time series. Through the classic functional principal component analysis of the sample variance operator, we obtain the eigenvalues and eigenfunctions which span a sample version of the dominant sub-space. Furthermore, we introduce a simple ratio criterion to consistently estimate the dimension of the dominant sub-space, and use a semiparametric local Whittle method to estimate the memory parameter. Monte-Carlo simulation studies are given to examine the finite-sample performance of the developed techniques.

Keywords: Fractional integration; functional principal component analysis; functional time series; local Whittle estimation; nonstationary process

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Comparing time varying regression quantiles under shift invariance

SUBHRA SANKAR DHAR[1,a] and WEICHI WU[2,b]

[1]*Department of Mathematics & Statistics, IIT Kanpur, Kanpur 208016, India,* [a]*subhra@iitk.ac.in*
[2]*Center for Statistical Science, Department of Industrial Engineering, Tsinghua University, 100084 Beijing, China,* [b]*wuweichi@mail.tsinghua.edu.cn*

This article investigates whether time-varying quantile regression curves are the same up to the horizontal shift or not. The errors and the covariates involved in the regression model are allowed to be locally stationary. We formalize this issue in a corresponding non-parametric hypothesis testing problem, and develop an integrated-squared-norm based test (SIT) as well as a simultaneous confidence band (SCB) approach. The asymptotic properties of SIT and SCB under null and local alternatives are derived. Moreover, the asymptotic properties of these tests are also studied when the compared data sets are dependent. We then propose valid wild bootstrap algorithms to implement SIT and SCB. Furthermore, the usefulness of the proposed methodology is illustrated via analysing simulated and real data related to COVID-19 outbreak.

Keywords: Bootstrap; comparison of curves; confidence band; hypothesis testing; locally stationary process; nonparametric quantile regression; COVID-19

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Loop-erased random walk branch of uniform spanning tree in topological polygons

MINGCHANG LIU[a] and HAO WU[b]

Tsinghua University, China, [a]*liumc_prob@163.com,* [b]*hao.wu.proba@gmail.com*

We consider uniform spanning tree (UST) in topological polygons with $2N$ marked points on the boundary with alternating boundary conditions. In an earlier work by Liu-Peltola-Wu, the authors derive the scaling limit of the Peano curve in the UST. They are variants of SLE_8. In this article, we derive the scaling limit of the loop-erased random walk branch (LERW) in the UST. They are variants of SLE_2. The conclusion is a generalization of an earlier work by Han-Liu-Wu where the authors derive the scaling limit of the LERW branch of UST when $N = 2$. When $N = 2$, the limiting law is $\mathrm{SLE}_2(-1,-1;-1,-1)$. However, the limiting law is no longer in the family of $\mathrm{SLE}_2(\rho)$ process as long as $N \geq 3$.

Keywords: Uniform spanning tree; loop-erased random walk; Schramm-Loewner evolution

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Bernoulli sums and Rényi entropy inequalities

MOKSHAY MADIMAN[1,a], JAMES MELBOURNE[2,b] and CYRIL ROBERTO[3,c]

[1]*University of Delaware, Department of Mathematical Sciences, 501 Ewing Hall, Newark DE 19716, USA,* [a]*madiman@udel.edu*

[2]*Centro de Investigación en Matemáticas, Probabilidad y Estadísticas, 36023 Guanajuato, Gto, México,* [b]*james.melbourne@cimat.mx*

[3]*MODAL'X, UMR CNRS 9023, FP2M, CNRS FR 2036, UPL, Université Paris Nanterre, 200 avenue de la République 92000 Nanterre, France,* [c]*croberto@math.cnrs.fr*

We investigate the Rényi entropy of sums of independent integer-valued random variables through Fourier theoretic means, and give sharp comparisons between the variance and the Rényi entropy for sums of independent Bernoulli random variables. As applications, we prove that a discrete "min-entropy power" is superadditive with respect to convolution modulo a universal constant, and give new bounds on an entropic generalization of the Littlewood-Offord problem that are sharp in the "Poisson regime".

Keywords: Rényi entropy; Littlewood-Offord; Poisson-binomial; entropy power; information theoretic inequalities; Concentration Functions

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Poisson approximation in χ^2 distance by the Stein-Chen approach

VYTAS ZACHAROVAS[a]

Institute of Computer Science, Vilnius University, Vilnius, Lithuania, [a]*vytas.zacharovas@mif.vu.lt*

The main purpose of the paper is to investigate the possibility of applying the Stein-Chen approach to estimate the χ^2 distance between Poisson distribution and Poisson binomial distribution. Earlier results concerning χ^2 distance between the above mentioned distributions either used analytical approach heavily based on the analysis of the generating functions or on rather lengthy and complicated elementary calculations. Applying the Stein-Chen approach we succeed in providing a very quick proof of upper bounds for χ^2 distance that are of comparable strength to the earlier estimates obtained by other approaches.

Keywords: Poisson apprximation; Charlier-Parseval identity; Charlier polynomials; χ^2 metric; the Stein-Chen approach

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Random normal matrices in the almost-circular regime

SUNG-SOO BYUN[1,a] and SEONG-MI SEO[2,b]

[1] *Center for Mathematical Challenges, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea,* [a] *sungsoobyun@kias.re.kr*
[2] *Department of Mathematics, Chungnam National University, Daejeon 34134, Republic of Korea,* [b] *smseo@cnu.ac.kr*

We study random normal matrix models whose eigenvalues tend to be distributed within a narrow "band" around the unit circle of width proportional to $1/n$, where n is the size of matrices. For general radially symmetric potentials with various boundary conditions, we derive the scaling limits of the correlation functions, some of which appear in the previous literature notably in the context of almost-Hermitian random matrices. We also obtain that fluctuations of the maximal and minimal modulus of the ensembles follow the Gumbel or exponential law depending on the boundary conditions.

Keywords: Almost-circular regime; maximal and minimal modulus; random normal matrix; scaling limits; universality

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Functional limit theorems for random walks perturbed by positive alpha-stable jumps

ALEXANDER IKSANOV[1,a], ANDREY PILIPENKO[2,3,b] and BEN POVAR[4,c]

[1]*Faculty of Computer Science and Cybernetics, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine,* [a]*iksan@univ.kiev.ua*
[2]*Department of the Theory of Stochastic Processes, Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine*
[3]*Department of Physics and Mathematics, National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine,* [b]*pilipenko.ay@gmail.com*
[4]*University of Warwick, Department of Statistics, Warwick, UK,* [c]*oleksandr.prykhodko@warwick.ac.uk*

Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables of zero mean and finite variance and $\eta_1, \eta_2, \dots$ positive i.i.d. random variables whose distribution belongs to the domain of attraction of an α-stable distribution, $\alpha \in (0,1)$. The two collections are assumed independent. We consider a Markov chain with jumps of two types. If the present position of the Markov chain is positive, then the jump ξ_k occurs; if the present position of the Markov chain is nonpositive, then the jump η_k occurs. We prove functional limit theorems for this and two closely related Markov chains under Donsker's scaling. The weak limit is a nonnegative process $(X(t))_{t\geq 0}$ satisfying a stochastic equation $\mathrm{d}X(t) = \mathrm{d}W(t) + \mathrm{d}U_\alpha(L_X^{(0)}(t))$, where W is a Brownian motion, U_α is an α-stable subordinator which is independent of W, and $L_X^{(0)}$ is a local time of X at 0. Also, we explain that X is a Feller Brownian motion with a 'jump-type' exit from 0.

Keywords: Feller Brownian motion; functional limit theorem; locally perturbed random walk; oscillating random walk

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On the rate of convergence of a deep recurrent neural network estimate in a regression problem with dependent data

MICHAEL KOHLER[1,a] and ADAM KRZYŻAK[2,b]

[1]*Fachbereich Mathematik, TU Darmstadt, Schlossgartenstr. 7, 64289 Darmstadt, Germany,* [a]*kohler@mathematik.tu-darmstadt.de*
[2]*Department of Computer Science and Software Engineering, Concordia University, 1455 De Maisonneuve Blvd. West, Montreal, Quebec, Canada H3G 1M8,* [b]*krzyzak@cs.concordia.ca*

A regression problem with dependent data is considered. Regularity assumptions on the dependency of the data are introduced, and it is shown that under suitable structural assumptions on the regression function a deep recurrent neural network estimate is able to circumvent the curse of dimensionality.

Keywords: Curse of dimensionality; recurrent neural networks; nonparametric regression estimation; rate of convergence

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On the singular values of complex matrix Brownian motion with a matrix drift

THEODOROS ASSIOTIS[a]

School of Mathematics, University of Edinburgh, James Clerk Maxwell Building, Peter Guthrie Tait Rd, Edinburgh EH9 3FD, UK, [a]*theo.assiotis@ed.ac.uk*

Let $\mathrm{Mat}_{\mathbb{C}}(K,N)$ be the space of $K \times N$ complex matrices. Let $\mathbf{B}_t$ be Brownian motion on $\mathrm{Mat}_{\mathbb{C}}(K,N)$ starting from the zero matrix and $\mathbf{M} \in \mathrm{Mat}_{\mathbb{C}}(K,N)$. We prove that, with $K \geq N$, the N eigenvalues of $(\mathbf{B}_t + t\mathbf{M})^* (\mathbf{B}_t + t\mathbf{M})$ form a Markov process with an explicit transition kernel. This generalizes a classical result of Rogers and Pitman (*Ann. Probab.* **9** (1981) 573–582) for multidimensional Brownian motion with drift which corresponds to $N = 1$. We then give two more descriptions for this Markov process. First, as independent squared Bessel diffusion processes in the wide sense, introduced by Watanabe (*Z. Wahrsch. Verw. Gebiete* **31** (1975) 115–124) and studied by Pitman and Yor (In *Stochastic Integrals (Proc. Sympos., Univ. Durham, Durham, 1980)* (1981) 285–370 Springer), conditioned to never intersect. Second, as the distribution of the top row of interacting squared Bessel type diffusions in some interlacing array. The last two descriptions also extend to a general class of one-dimensional diffusions.

Keywords: Singular values; matrix Brownian motion with drift; squared Bessel diffusions in wide sense; non-intersecting paths; interacting diffusions; interlacing arrays

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A note on one-dimensional Poincaré inequalities by Stein-type integration

GILLES GERMAIN[a] and YVIK SWAN[b]

Université libre de Bruxelles, Département de Mathématique, Campus Plaine, Boulevard du Triomphe CP210, B-1050 Brussels, [a]*gilles.germain@ulb.be,* [b]*yvik.swan@ulb.be*

We study the weighted Poincaré constant $C(p,w)$ of a probability density p with weight function w using integration methods inspired by Stein's method. We obtain a new version of the Chen-Wang variational formula which, as a byproduct, yields simple upper and lower bounds on $C(p,w)$ in terms of the so-called Stein kernel of p. We also iterate these variational formulas so as to build sequences of nested intervals containing the Poincaré constant, sequences of functions converging to said constant, as well as sequences of functions converging to the solutions of the corresponding spectral problem. Our results rely on the properties of a pseudo inverse operator of the classical Sturm-Liouville operator. We illustrate our methods on a variety of examples: Gaussian functionals, weighted Gaussian, beta, gamma, Subbotin, and Weibull distributions.

Keywords: Stein operators; Poincaré inequalities; Chen-Wang variational formula

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