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Aims and Scope

Issued four times per year, BERNOULLI is the flagship journal of the Bernoulli Society for Mathematical Statistics and Probability. The journal aims at publishing original research contributions of the highest quality in all subfields of Mathematical Statistics and Probability. The main emphasis of Bernoulli is on theoretical work, yet discussion of interesting applications in relation to the proposed methodology is also welcome.

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Treatment effect estimation with efficient data aggregation

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Data aggregation, also known as meta analysis, is widely used to combine knowledge on parameters shared in common (e.g., average treatment effect) between multiple studies. In this paper, we introduce an attractive data aggregation scheme that pools summary statistics from various existing studies. Our scheme informs the design of new validation studies and yields unbiased estimators for the shared parameters. In our setup, each existing study applies a LASSO regression to select a parsimonious model from a large set of covariates. It is well known that post-hoc estimators, in the selected model, tend to be biased. We show that a novel technique called *data carving* yields us a new unbiased estimator by aggregating simple summary statistics from all existing studies. The proposed estimator has two key features: (a) we make the fullest possible use of data, from all studies, without the risk of bias from model selection; (b) we enjoy the added benefit of individual data privacy, because raw data from these studies need not be shared or stored for efficient estimation.

Keywords: Conditional inference; debiased estimation; data carving; LASSO; meta analysis

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Varying coefficient regression: Revisit and parametric help

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This paper concerns the estimation of varying coefficient models, which is considered very useful in analyzing the regression relationship between variables. The purpose of this paper is threefold. Firstly, we introduce a new formulation of varying coefficient regression under which a structure-respecting constraint is developed for identifying the model components. Secondly, we develop a full account of locally linear kernel smoothing approach to estimating varying coefficient models which is largely missing in the literature. Thirdly, we address a bias reduction technique applied to locally linear varying coefficient regression which turns out to be successful under mild condition. We develop our methodology and theory for response variables taking values in a general Hilbert space. We discuss relevant theory for the associated projection operators, the convergence of an iterative backfitting algorithm, and the error rates and asymptotic distributions of the estimators. We also include some simulation results demonstrating the success of the proposed approach.

Keywords: Additive model; Hilbert space; local linear smoothing; non-Euclidean data; smooth backfitting; varying coefficient model

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Almost sure growth of integrated supOU processes

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Superpositions of Ornstein-Uhlenbeck processes allow a flexible dependence structure, including long range dependence for OU-type processes. Their complex asymptotics are governed by three effects: the behavior of the Lévy measure both at infinity and at zero, and the behavior at zero of the measure governing the dependence. We establish almost sure rates of growth depending on the characteristics of the process and prove a Marcinkiewicz–Zygmund type SLLN for the integrated process.

Keywords: Almost sure properties; infinitely divisible random measure; Marcinkiewicz–Zygmund type strong law of large numbers; rate of growth; supOU processes

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Ergodicity, CLT and asymptotic maximum of the Airy_1 process

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We first show that the Airy_1 process is associated using the association property of the solution to the stochastic heat equation and convergence of the KPZ equation to the KPZ fixed point. Then we apply Newman's inequality to establish the ergodicity and central limit theorem for the Airy_1 process. Combined with the asymptotic behavior of the tail probability, we derive a Poisson limit theorem for the Airy_1 process and give a precise estimate on the asymptotic behavior of the maximum of the Airy_1 process over an interval. Analogous results for the Airy_2 process are also presented.

Keywords: Airy_1 process; association; CLT; ergodicity

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Finite moments testing in a general class of nonlinear time series models

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We investigate the problem of testing the finiteness of moments for a class of semi-parametric time series encompassing many commonly used specifications. The existence of positive-power moments of the strictly stationary solution is characterized by the Moment Determining Function (MDF) of the model, which depends on the parameter driving the dynamics and on the distribution of the innovations. We establish the asymptotic distribution of the empirical MDF, from which tests of moments are deduced. Alternative tests based on the estimation of the Maximal Moment Exponent (MME) are studied. Power comparisons based on local alternatives and the Bahadur approach are proposed. We provide an illustration on real financial data and show that semi-parametric estimation of the MME provides an interesting alternative to Hill's nonparametric estimator of the tail index.

Keywords: Efficiency comparisons of tests; maximal moment exponent; stochastic recurrence equation; tail index

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A central limit theorem for a sequence of conditionally centered random fields

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A central limit theorem is established for a sum of random variables belonging to a sequence of random fields. The fields are assumed to have zero mean conditional on the past history and to satisfy certain conditional α-mixing conditions in space or time. Exploiting conditional centering and the space-time structure, the limiting normal distribution is obtained for increasing spatial domain, increasing length of the sequence, or both of these. The theorem is very well suited for establishing asymptotic normality in the context of unbiased estimating function inference for a wide range of space-time processes. This is pertinent given the abundance of space-time data. Two examples demonstrate the applicability of the theorem.

Keywords: Central limit theorem; conditional α-mixing; conditional centering; estimating function; random field; space-time; spatio-temporal; vector autoregression

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Nearly minimax robust estimator of the mean vector by iterative spectral dimension reduction

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We study the problem of robust estimation of the mean vector of a sub-Gaussian distribution. We introduce an estimator based on spectral dimension reduction (SDR) and establish a finite sample upper bound on its error that is minimax-optimal up to a logarithmic factor. Furthermore, we prove that the breakdown point of the SDR estimator is equal to $1/2$, the highest possible value of the breakdown point. In addition, the SDR estimator is equivariant by similarity transforms and has low computational complexity. More precisely, in the case of n vectors of dimension p—at most εn out of which are adversarially corrupted—the SDR estimator has a squared error of order $(\mathbf{r}_\Sigma/n + \varepsilon^2 \log(1/\varepsilon))\log p$ and a running time of order $p^3 + np^2$. Here, $\mathbf{r}_\Sigma \le p$ is the effective rank of the covariance matrix of the reference distribution. Another advantage of the SDR estimator is that it does not require knowledge of the contamination rate and does not involve sample splitting. We also investigate extensions of the proposed algorithm and of the obtained results in the case of (partially) unknown covariance matrix.

Keywords: Breakdown point; minimax optimality; robustness; spectral method

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The extended Ville's inequality for nonintegrable nonnegative supermartingales

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Following the initial work by Robbins, we rigorously present an extended theory of nonnegative supermartingales, requiring neither integrability nor finiteness. In particular, we derive a key maximal inequality foreshadowed by Robbins, which we call the extended Ville's inequality, that strengthens the classical Ville's inequality (for integrable nonnegative supermartingales), and also applies to our nonintegrable setting. We derive an extension of the method of mixtures, which applies to σ-finite mixtures of our extended nonnegative supermartingales. We present some implications of our theory for sequential statistics, such as the use of *improper* mixtures (priors) in deriving nonparametric confidence sequences and (extended) e-processes.

Keywords: Confidence sequences; improper priors; non-integrable processes; martingales; maximal inequalities; sequential statistics

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Local goodness-of-fit testing for Hölder-continuous densities: Minimax rates

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We consider the goodness-of-fit testing problem for Hölder smooth densities over $\mathbb{R}^d$: given n iid observations with unknown density p and given a known density p_0, we investigate how large ρ should be to distinguish, with high probability, the case $p = p_0$ from the composite alternative of all Hölder-smooth densities p such that $\|p - p_0\|_t \geq \rho$ where $t \in [1, 2]$. The densities are assumed to be defined over $\mathbb{R}^d$ and to have Hölder smoothness parameter $\alpha > 0$. In the present work, we solve the case $\alpha \leq 1$ and handle the case $\alpha > 1$ using an additional technical restriction on the densities. We identify matching upper and lower bounds on the local minimax rates of testing, given explicitly in terms of p_0. We propose novel test statistics which we believe could be of independent interest. We also establish the first definition of an explicit cutoff u_B allowing us to split $\mathbb{R}^d$ into a bulk part (defined as the subset of $\mathbb{R}^d$ where p_0 takes only values greater than or equal to u_B) and a tail part (defined as the complementary of the bulk), each part involving fundamentally different contributions to the local minimax rates of testing.

Keywords: Hölder-continuous densities; hypothesis testing; minimax separation radius

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Weak equivalence of local independence graphs

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Classical graphical modeling of random vectors uses graphs to encode conditional independence. In graphical modeling of multivariate stochastic processes, graphs may encode so-called local independence analogously. If some coordinate processes of the multivariate stochastic process are unobserved, the local independence graph of the observed coordinate processes is a directed mixed graph (DMG). Two DMGs may encode the same local independences in which case we say that they are Markov equivalent. Markov equivalence is a central notion in graphical modeling. We show that deciding Markov equivalence of DMGs is coNP-complete, even under a sparsity assumption. As a remedy, we introduce *weak* equivalence relations on DMGs that are less granular than Markov equivalence. This leads to feasible algorithms for naturally occurring computational problems, and the equivalence classes of a weak equivalence relation have attractive properties. In particular, each equivalence class has a greatest element which allows a concise representation of an equivalence class. Moreover, these equivalence relations define a hierarchy of granularity in the graphical modeling which leads to simple and interpretable connections between equivalence classes corresponding to different levels of granularity. Weak equivalence is an approximation to the more expressive Markov equivalence, and we also describe some properties of this approximation.

Keywords: Directed mixed graphs; local independence; local independence graph; Markov equivalence; weak equivalence; μ-separation

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Viscosity estimation for 2D pipe flows I. Construction, consistency, asymptotic normality

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We consider the motion of incompressible viscous fluid in a rectangle, imposing the periodicity condition in one direction and the no-slip boundary condition in the other. Assuming that the flow is subject to an external random force, white in time and regular in space, we construct an estimator $\hat{\nu}_t$ for the viscosity ν using only observations of the L^2 norm of the vorticity on the time interval $[0,t]$. The goal of the paper is to investigate the asymptotic properties of $\hat{\nu}_t$ as $t \to +\infty$. It is proved that the estimator $\hat{\nu}_t$ is strongly consistent and asymptotically normal. The proof of consistency is based on the explicit formula for the estimator and some bounds for trajectories, while that of asymptotic normality uses in addition mixing properties of the Navier–Stokes flow.

Keywords: Asymptotic normality; consistency; Navier–Stokes equations; viscosity estimation; white noise

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PEBBLE: A second order correct bootstrap method in logistic regression

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Logistic regression is often used in many different fields, such as clinical trials, biomedical surveys, marketing, and banking, to predict the probability of a binary outcome. In this paper we propose a novel Bootstrap technique for approximating the distribution of the maximum likelihood estimator (MLE) of the regression parameter vector. Improved inference performance is obtained over the traditional normal approximation via establishing second order correctness. The main challenge in establishing second order correctness remains in the fact that the response variable being binary, the MLE may have a lattice structure resulting in non-existence of formal Edgeworth expansion. In order to achieve the second order correctness, a smoothing technique developed in Lahiri (*J. Multivariate Anal.* **45** (1993) 247–256) is adopted to define the original and Bootstrapped studentized pivots. Second order results are also extended to the one-dimensional smooth functions of the regression parameter e.g., odds ratio and success probability. Simulation experiments are performed to evaluate the finite-sample properties of the proposed Bootstrap method. The proposed methodology is demonstrated with an application on real dataset in healthcare operations.

Keywords: Lattice; logistic regression; perturbation Bootstrap; smoothing; SOC

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Nonparametric estimation for additive concurrent regression models

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Consider an additive functional regression model where a one-dimensional response process $Y(t)$ and a K-dimensional explanatory random process $X_j(t)$, $j = 1, \dots, K$, are observed for $t \in [0, \tau]$, with fixed τ. Examples of such explanatory processes are continuous or inhomogeneous counting processes. The linear coefficients of the model are K unknown deterministic functions $t \mapsto b_j(t)$, $j = 1, \dots, K$, $t \in [0, \tau]$. From N independent trajectories, we build a nonparametric least-squares estimator $(\hat{b}_1, \dots, \hat{b}_K)$ of $(b_1, \dots, b_K)$, where each $\hat{b}_j$, $1 \le j \le K$, is given by its expansion on a finite-dimensional space. We prove a bound on the mean-square risk of the estimator, from which rates of convergence are obtained and are established to be optimal. An adaptive procedure, achieving simultaneous and anisotropic selection of each space dimension, is then tailored and an oracle risk bound is proved. The procedure is studied numerically and implemented on a real dataset of electric consumption.

Keywords: Adaptive estimation; continuous observation; functional data; least-squares estimator; nonparametric regression function estimation; projection method

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Efficient quantile regression under censoring using Laguerre polynomials

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In this paper, we consider a novel methodology to estimate linear quantile regression models when the response is randomly right censored. The proposed methodology is based on approximating the error distribution by means of an extension of the Laplace distribution, that is flexible enough to approximate any continuous distribution as long as the number of parameters in this Enriched Laplace distribution grows to infinity. The extension is obtained by enriching the Laplace density by means of Laguerre polynomials in such a way that the new density has by construction the property that its quantile of interest is equal to zero. Hence, when the error term has an Enriched Laplace density, it satisfies by construction the constraints of a quantile regression model. We will show that, with this new quantile regression model, we can obtain novel estimators of the quantile function, which are shown to be consistent and asymptotically normal. We also establish the asymptotic efficiency bound, and show by means of a simulation study and the analysis of data on Covid-19 patients that the proposed method works well in practice compared to competing estimators.

Keywords: Censored data; Laguerre polynomials; quantile regression; semi-parametric regression

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On non-negative solutions of stochastic Volterra equations with jumps and non-Lipschitz coefficients

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We consider one-dimensional stochastic Volterra equations with jumps for which we establish conditions upon the convolution kernel and coefficients for the strong existence and pathwise uniqueness of a non-negative càdlàg solution. By using the approach recently developed by (*Stochastic Process. Appl.* **181** (2025) Paper No. 104535), we show the strong existence by using a nonnegative approximation of the equation whose convergence is proved via a variant of the Yamada–Watanabe approximation technique. We apply our results to Lévy-driven stochastic Volterra equations. In particular, we are able to define a Volterra extension of the so-called *alpha-stable Cox–Ingersoll–Ross process*, which is especially used for applications in Mathematical Finance.

Keywords: Affine Volterra processes; alpha-stable Lévy process; pathwise uniqueness; stochastic Volterra equations with jumps; strong solution

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Log-concave density estimation in undirected graphical models

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We study the problem of maximum likelihood estimation of densities that are log-concave and lie in the graphical model corresponding to a given undirected graph G. More precisely, we assume that each density in our family factorizes according to the graph G and all factors are log-concave. We show that the maximum likelihood estimate (MLE) is the product of the exponentials of several tent functions, one for each maximal clique of G. While the set of log-concave densities in a graphical model is infinite-dimensional, our results imply that the MLE can be found by solving a finite-dimensional convex optimization problem. We provide an implementation and a few examples. Furthermore, we show that the MLE exists and is unique with probability 1 as long as the number of sample points is larger than the size of the largest clique of G when G is chordal. We show that the MLE is consistent when the graph G is a disjoint union of cliques. Finally, we discuss the conditions under which a log-concave density in the graphical model of G has a log-concave factorization according to G.

Keywords: Chordal graphs; convex decomposition of functions; graphical models; log-concave density estimation; maximum likelihood estimation

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Non-parametric estimates for graphon mean-field particle systems

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We consider the graphon mean-field system introduced in (*Ann. Appl. Probab.* **33** (2023) 3587–3619). It is the large-population limit of a heterogeneously interacting diffusive particle system, where the interaction is of mean-field type with weights characterized by an underlying graphon function. Through observation of continuous-time trajectories within the particle system, we construct plug-in estimators of the particle density, the drift coefficient, and thus the graphon edge weight of the mean-field system. Our estimators for the density and drift are direct results of kernel interpolation on the empirical data, and a deconvolution method leads to an estimator of the underlying graphon function. We show that, as the number of particles increases, the estimated edge weight converges to the true graphon function pointwisely and, consequently, in the cut metric. Besides, we conduct a minimax analysis within a particular class of particle systems to justify the pointwise optimality of the density and drift estimators.

Keywords: Graphon mean-field system; interacting particles; kernel estimation; minimax analysis

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Simultaneous semiparametric inference for single-index models

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In the common partially linear single-index model we establish a Bahadur representation for a smoothing spline estimator of all model parameters and use this result to prove the joint weak convergence of the estimator of the index link function at a given point, together with the estimators of the parametric regression coefficients. We obtain the surprising result that, despite of the nature of single-index models where the link function is evaluated at a linear combination of the index-coefficients, the estimator of the link function and the estimator of the index-coefficients are asymptotically independent. Our approach leverages a delicate analysis based on reproducing kernel Hilbert space and empirical process theory. We show that the smoothing spline estimator achieves the minimax optimal rate with respect to the L^2-risk and consider several statistical applications where joint inference on all model parameters is of interest. In particular, we develop a simultaneous confidence band for the link function and propose inference tools to investigate if the maximum absolute deviation between the (unknown) link function and a given function exceeds a given threshold. We also construct tests for joint hypotheses regarding model parameters which involve both the nonparametric and parametric components and propose novel multiplier bootstrap procedures to avoid the estimation of unknown asymptotic quantities.

Keywords: Bahadur representation; efficient estimators; joint estimating equation; M-estimation; multiplier bootstrap; relevant hypotheses; reproducing kernel Hilbert space; semiparametric regression; single-index model

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Data fusion methods for the heterogeneity of treatment effect and confounding function

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The heterogeneity of treatment effect (HTE) lies at the heart of precision medicine. Randomized controlled trials are gold-standard for treatment effect estimation but are typically underpowered for heterogeneous effects. In contrast, large observational studies have high predictive power but are often confounded due to the lack of randomization of treatment. We show that the observational study, even subject to hidden confounding, may be used to empower trials in estimating the HTE using the notion of confounding function. The confounding function summarizes the impact of unmeasured confounders on the difference between the observed treatment effect and the causal treatment effect, given the observed covariates, which is unidentifiable based only on the observational study. Coupling the trial and observational study, we show that the HTE and confounding function are identifiable. We then derive the semiparametric efficient scores and the integrative estimators of the HTE and confounding function. We clarify the conditions under which the integrative estimator of the HTE is strictly more efficient than the trial estimator. Finally, we illustrate the integrative estimators via simulation and an application.

Keywords: Estimating equation; goodness of fit; over-identification test; semiparametric efficiency; structural model

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Sequential kernel embedding for mediated and time-varying dose response curves

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We propose simple nonparametric estimators for mediated and time-varying dose response curves based on kernel ridge regression. By embedding Pearl's mediation formula and Robins' g-formula with kernels, we allow treatments, mediators, and covariates to be continuous in general spaces, and also allow for nonlinear treatment-confounder feedback. Our key innovation is a reproducing kernel Hilbert space technique called sequential kernel embedding, which we use to construct simple estimators that account for complex feedback. Our estimators preserve the generality of classic identification while also achieving nonasymptotic uniform rates. In nonlinear simulations with many covariates, we demonstrate strong performance. We estimate mediated and time-varying dose response curves of the US Job Corps, and clean data that may serve as a benchmark in future work. We extend our results to mediated and time-varying treatment effects and counterfactual distributions, verifying semiparametric efficiency and weak convergence.

Keywords: Continuous treatment; reproducing kernel Hilbert space; treatment-confounder feedback

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Pattern-based tests for two-dimensional copulas

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In statistics permutations typically arise in the context of rank plots for two-dimensional data. Such plots can also be interpreted as discrete copulas. In discrete mathematics, typically in the context of the description of large (non-random) objects, two-dimensional copulas appear as limits of permutations and are then known as permutons if the topology refers to the convergence of pattern frequencies. We obtain a functional central limit theorem for such pattern frequencies in the context of two-dimensional random samples. The result serves as the basis for nonparametric goodness-of-fit tests, for two-sample tests, and for tests of symmetry. This includes a suitable variant of the bootstrap for obtaining critical values. Pattern-based procedures are also of interest in a parametric context. We consider two examples, the Farlie-Gumbel-Morgenstern class and a family of delay copulas. We discuss implementation aspects of the resulting procedures and we provide a simulation study that supplements the theoretical results in the nonparametric case.

Keywords: Bootstrap; copula; Cramér-von Mises type test; goodness-of-fit test; Kolmogorov-Smirnov type test; permuton; random permutation; rank plot; two-sample test

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Parabolic Anderson model with colored noise on the torus

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We construct an intrinsic family of Gaussian noises on the d-dimensional flat torus $\mathbb{T}^d$. It is the analogue of the colored noise on $\mathbb{R}^d$ and allows us to study stochastic PDEs on the torus in the Itô sense in high dimensions. With this noise, we consider the *parabolic Anderson model* (PAM) with measure-valued initial conditions and establish some basic properties of the solution, including a sharp upper and lower bound for the moments and Hölder continuity in space and time. The study of the toy model of $\mathbb{T}^d$ in the present paper is a first step in our effort to understand how geometry and topology play a role in the behavior of stochastic PDEs on general (compact) manifolds.

Keywords: Brownian bridge; Dalang's condition; intermittency; moment asymptotics; measure-valued initial condition; moment Lyapunov exponent; stochastic heat equation on torus; theta function

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Subexponential estimates for the first hitting time of a Brownian motion with singular drift

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We study the effect of a power law drift on the Brownian motion in the positive half-line, where the order of the drift at 0 and ∞ is different. The first hitting time of 0 is finite almost surely, even when the drift near 0 is positive and unbounded. We obtain subexponential estimates for the tail distribution of the hitting time of 0, that are independent of the singular behavior of the drift near the origin.

Keywords: Brownian motion; Brownian motion with drift; half-line; h-transform; large deviations; stochastic comparison

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Compound multivariate Hawkes processes: Large deviations and rare event simulation

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In this paper, we establish a large deviations principle for a multivariate compound process induced by a multivariate Hawkes process with random marks. Our proof hinges on showing essential smoothness of the limiting cumulant of the multivariate compound process, resolving the inherent complication that this cumulant is implicitly characterized through a fixed-point representation. We employ the large deviations principle to derive logarithmic asymptotic results on the marginal ruin probabilities of the associated multivariate risk process. We also show how to conduct rare event simulation in this multivariate setting using importance sampling and prove the asymptotic efficiency of our importance sampling based estimators.

Keywords: Compound processes; Hawkes processes; large deviation; mutual excitation; rare event simulation; ruin probability

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On numerical discretizations that preserve probabilistic limit behaviors for time-homogeneous Markov processes

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For the general time-homogeneous Markov process, do numerical discretizations exactly preserve its probabilistic limit behaviors, in particular the strong LLN and the CLT? This paper gives a positive answer to the question by proposing a unified and transparent approach to investigating probabilistic limit behaviors of numerical discretizations of time-homogeneous Markov processes. Once the properties of uniformly mixing and convergence for numerical discretizations are satisfied, it is shown that the time-averages of numerical discretizations converge to the ergodic limit in the almost surely sense and that the normalized time-averages converge in distribution to a normal distribution. The limits coincide with the ones for the underlying Markov process. Our results have the merit of the flexible application to numerical discretizations for a large class of stochastic differential equations. Notably, the preservation of these probabilistic limit behaviors of the full discretization of the stochastic Allen–Cahn equation and numerical discretizations of stochastic functional differential equations are obtained for the first time.

Keywords: Central limit theorem; Markov process; numerical discretization; probabilistic limit behaviors; strong law of large numbers

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Information geometry and asymptotics for Kronecker covariances

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We explore the information geometry and asymptotic behaviour of estimators for Kronecker-structured covariances, in both growing-n and growing-p scenarios, with a focus towards examining the estimator proposed by Linton and Tang, which we refer to as the partial trace estimator. It is shown that the partial trace estimator is asymptotically inefficient. An explanation for this inefficiency is that the partial trace estimator does not scale sub-blocks of the sample covariance matrix optimally. To correct for this, an asymptotically efficient, rescaled partial trace estimator is introduced. Motivated by this rescaling, we introduce an orthogonal parameterization for the set of Kronecker covariances. High-dimensional consistency results using the partial trace estimator are obtained that demonstrate a blessing of dimensionality. In settings where an array has at least order three, it is shown that as the array dimensions jointly increase, it is possible to consistently estimate the Kronecker covariance matrix, even when the sample size is one.

Keywords: Exponential family; Fisher information metric; high-dimensional; Kronecker product; orthogonal parameterization; partial trace; tensor

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Prediction of random variables by excursion metric projections

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Dedicated to the memory of Dr Vitalii Makogin who unexpectedly passed away on the 8th of May 2024

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We use the concept of excursions to predict random variables without moment existence assumptions. To do so, an excursion metric on the space of random variables is defined as a weighted L^1-distance. Using equivalent forms of this metric and the specific choice of excursion levels, we formulate the prediction problem as a minimization of a certain target functional. Existence of the solution and weak consistency of the predictor are discussed. An application to extrapolating stationary heavy-tailed random functions illustrates the use of our approach. Numerical experiments predicting Gaussian, α-stable and further heavy–tailed time series complement the paper.

Keywords: Excursion; extrapolation; forecasting; Gini metric; heavy tails; level set; (linear) prediction; stable random function; stationary random field; statistical learning; time series

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Semi-parametric goodness-of-fit testing for INAR models

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Among the various models designed for dependent count data, integer-valued autoregressive (INAR) processes enjoy great popularity. Typically, statistical inference for INAR models uses asymptotic theory that relies on rather stringent (parametric) assumptions on the innovations such as Poisson or negative binomial distributions. In this paper, we present a novel semi-parametric goodness-of-fit test tailored for the INAR model class. Relying on the INAR-specific shape of the joint probability generating function, our approach allows for model validation of INAR models without specifying the (family of the) innovation distribution. We derive the limiting null distribution of our proposed test statistic, prove consistency under fixed alternatives and discuss its asymptotic behavior under local alternatives. By manifold Monte Carlo simulations, we illustrate the overall good performance of our testing procedure in terms of power and size properties. In particular, it turns out that the power can be considerably improved by using higher-order test statistics. In supplementary material, we provide an application to three real-world economic data sets.

Keywords: Bootstrap; count time series; goodness-of-fit; local power; probability generating function; semi-parametric estimation

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Asymptotic expansions in the central limit theorem for a super-Brownian motion

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Let $\{X_t\}_{t\geq 0}$ be a d-dimensional supercritical super-Brownian motion started from the Dirac measure at zero. In this work, we prove that, almost surely, for any bounded measurable function f,

$$\lim_{t\to\infty} e^{-\beta t} X_t(f(\cdot/\sqrt{t})) = \nu(f)W,$$

where ν is the standard Gaussian measure, β is a constant satisfying $e^{\beta t} = \mathbb{E}[X_t(\mathbb{R}^d)]$ and W is some non-negative random variable. Moreover, we establish the asymptotic expansion of any order in above central limit theorem.

Keywords: Fourier transform; Garsia's inequality; super-Brownian motion

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Estimating the history of a random recursive tree

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This paper studies the problem of estimating the order of arrival of the vertices in a random recursive tree. Specifically, we study two fundamental models: the uniform attachment model and the linear preferential attachment model. We propose an order estimator based on the Jordan centrality measure and define a family of risk measures to quantify the quality of the ordering procedure. Moreover, we establish a minimax lower bound for this problem, and prove that the proposed estimator is nearly optimal. Finally, we numerically demonstrate that the proposed estimator outperforms degree-based and spectral ordering procedures.

Keywords: Latent variable estimation; network archaeology; seriation

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High-dimensional partially linear additive models on Riemannian manifolds

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We develop partially linear additive modeling techniques for a response variable situated in a Riemannian manifold, where the number of linear components may diverge with the sample size. We explore both ridge-based and SCAD-based methods for estimating the linear coefficients, and modify the smooth backfitting techniques to estimate the nonparametric components. We show that our proposed variable selection approaches exhibit selection consistency and that the parametric estimators possess the oracle property. Moreover, our nonparametric estimators achieve the convergence rate of univariate nonparametric models. The advantages of the proposed method are demonstrated through numerical studies and an application to an ADNI dataset.

Keywords: Additive models; high-dimensional covariate; partially linear models; profiling; smooth backfitting; SPD matrix data

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Hypothesis testing for functional linear models via bootstrapping

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Hypothesis testing for the slope function in functional linear regression is of both practical and theoretical interest. We develop a novel test for the nullity of the slope function, where testing the slope function is transformed into testing a high-dimensional vector based on functional principal component analysis. This transformation fully circumvents ill-posedness in functional linear regression, thereby enhancing numeric stability. The proposed method leverages the technique of bootstrapping max statistics and exploits the inherent variance decay property of functional data, improving the empirical power of tests especially when the sample size is limited or the signal is relatively weak. We establish validity and consistency of our proposed test when the functional principal components are derived from data. Moreover, we show that the test maintains its asymptotic validity and consistency, even when including *all* empirical functional principal components in our test statistics. This sharply contrasts with the task of estimating the slope function, which requires a delicate choice of the number (at most in the order of $\sqrt{n}$) of functional principal components to ensure estimation consistency. This distinction highlights an interesting difference between estimation and statistical inference regarding the slope function in functional linear regression. To the best of our knowledge, the proposed test is the first of its kind to utilize all empirical functional principal components in functional linear models.

Keywords: Ill-posedness; max statistic; slope function; uniform bootstrap approximation; uniform Gaussian approximation

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A new non-parametric Kendall's tau for matrix-valued elliptical observations

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In this article, we propose a new non-parametric Kendall's τ for matrix-variate observations that are ubiquitous in areas such as finance and medical imaging, named as row/column matrix Kendall's tau. For a random matrix following the matrix-variate elliptically contoured distribution, we show that the eigenspaces of the proposed row/column matrix Kendall's tau coincide with those of the row/column scatter matrix respectively, with the same descending order of the eigenvalues. To show the usefulness of the new non-parametric Kendall's τ, we focus on a two-way dimension reduction model, namely the growing popular "matrix factor model" in the literature. We perform eigenvalue decomposition to the generalized row/column matrix Kendall's tau for recovering the loading spaces of the matrix factor model. We also propose to estimate the pair of factor numbers by exploiting the eigenvalue-ratios of the row/column matrix Kendall's tau. Theoretically, we derive the convergence rates of the estimators for loading spaces, factor scores and common components without any moment constraints on the idiosyncratic errors. Bahadur representation for the estimated loadings is also provided. The proposed methods can further be generalized to analyze high-order tensors. Thorough simulation studies are conducted to show the higher degree of robustness of the proposed estimators over the existing ones. Analysis of a financial dataset of asset returns in the supplement illustrates the empirical usefulness of the proposed method. An R package named "MKendall" to implement the procedure is available on R CRAN.

Keywords: Elliptical distribution; Kendall's tau; matrix factor model; principal component analysis

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Monitoring of functional time series

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Detecting changes in dependent time series is an important topic in statistics. Tests for change points have been developed for different data structures, including multivariate, high-dimensional and even functional data. In functional data analysis, most literature focuses on retrospective settings, where the full dataset is collected before searching for changes. In this work, we follow the different paradigm of sequential monitoring. Here, data arrive successively, and the user tries to determine on an ongoing basis whether a change has already occurred - for an open-ended time period. This problem has not been investigated before for infinite dimensional data, where the standard tools to validate monitoring schemes are not available. In this work, we propose a different approach, combining strong approximations for Banach space valued data, with bounds from empirical process theory. Therewith, we validate monitoring schemes on a range of function spaces, including L^p-functions and spaces of continuous functions. In the latter case, we also propose theory for data with an unbounded domain - a problem rarely considered for functional data. An application of our theory to monitoring sequences of estimated densities is given. We investigate practical aspects of our approach in a simulation study and a data example on Exchange Traded Funds.

Keywords: Banach space; change point; functional data; monitoring

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