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On the critical probability of percolation on random causal triangulations

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Abstract. In this work, we study bond percolation on random causal triangulations. While in the sub-critical regime there is no phase transition, we show that for percolation on critical random causal triangulations there exists a non-trivial phase transition and we compute an upper bound for the critical probability. Furthermore, the critical value is shown to be almost surely constant.

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Key words and phrases. Percolation, causal triangulations, phase transition.

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A generating function approach to branching random walks

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Abstract. It is well known that the behaviour of a branching process is completely described by the generating function of the offspring law and its fixed points. Branching random walks are a natural generalization of branching processes: a branching process can be seen as a one-dimensional branching random walk. We define a multidimensional generating function associated to a given branching random walk. The present paper investigates the similarities and the differences of the generating functions, their fixed points and the implications on the underlying stochastic process, between the one-dimensional (branching process) and the multidimensional case (branching random walk). In particular, we show that the generating function of a branching random walk can have uncountably many fixed points and a fixed point may not be an extinction probability, even in the irreducible case (extinction probabilities are always fixed points). Moreover, the generating function might not be a convex function. We also study how the behaviour of a branching random walk is affected by local modifications of the process. As a corollary, we describe a general procedure with which we can modify a continuous-time branching random walk which has a weak phase and turn it into a continuous-time branching random walk which has strong local survival for large or small values of the parameter and non-strong local survival for intermediate values of the parameter.

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Key words and phrases. Branching random walk, branching process, strong local survival, generating function, fixed point, extinction probability.

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Improved asymptotic estimates for the contact process with stirring

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Abstract. We study the contact process with stirring on $\mathbb{Z}^d$. In this process, particles occupy vertices of $\mathbb{Z}^d$; each particle dies with rate 1 and generates a new particle at a randomly chosen neighboring vertex with rate λ, provided the chosen vertex is empty. Additionally, particles move according to a symmetric exclusion process with rate N. For any d and N, there exists λ_c such that, when the system starts from a single particle, particles go extinct when $\lambda < \lambda_c$ and have a chance of being present for all times when $\lambda > \lambda_c$. Durrett and Neuhauser proved that λ_c converges to 1 as N goes to infinity, and Konno, Katori and Berezin and Mytnik obtained dimension-dependent asymptotics for this convergence, which are sharp in dimensions 3 and higher. We obtain a lower bound which is new in dimension 2 and also gives the sharp asymptotics in dimensions 3 and higher. Our proof involves an estimate for two-type renewal processes which is of independent interest.

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Key words and phrases. Interacting particle systems, contact process, contact process with rapid stirring.

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On bivariate inverse Weibull distribution

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Abstract. Inverse Weibull distribution has been used quite successfully to analyze lifetime data which has non-monotone hazard function. The main aim of this paper is to introduce bivariate inverse Weibull distribution along the same line as the Marshall–Olkin bivariate exponential distribution, so that the marginals have inverse Weibull distributions. The proposed bivariate inverse Weibull distribution has four parameters and it has a singular component. Therefore, it can be used quite successfully if there are ties in the data. The joint probability density function, the joint cumulative distribution function and the joint survival function are all in closed forms. Several properties of this distribution have been discussed. It is observed that the proposed distribution can be obtained from the Marshall–Olkin copula. The maximum likelihood estimators of the unknown parameters cannot be obtained in closed form, and we propose to use EM algorithm to compute the maximum likelihood estimators. We propose to use parametric bootstrap method for construction of confidence intervals of the different parameters. We present some simulation experiments results to show the performances of the EM algorithm and they are quite satisfactory. We provide the Bayesian analysis of the unknown parameters based on very flexible priors. We analyze one bivariate American Football League data set for illustrative purposes, and it is observed that this model provides a slightly better fit than some of the existing models. Finally, we present some generalization to the multivariate case.

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Key words and phrases. Marshall–Olkin bivariate exponential distribution, maximum likelihood estimator, failure rate, EM algorithm, Fisher information matrix.

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On estimating the scale parameter of the selected uniform population under the entropy loss function

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Abstract. Let $\pi_1, \dots, \pi_k$ be k (≥ 2) independent populations, where π_i denotes the uniform distribution over the interval $(0, \theta_i)$ and $\theta_i > 0$ $(i = 1, \dots, k)$ is an unknown scale parameter. Let $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ be the ordered values of $\theta_1, \dots, \theta_k$. The population $\pi_{(k)}$ $(\pi_{(1)})$ associated with the unknown parameter $\theta_{[k]}$ $(\theta_{[1]})$ is called the best (worst) population. For selecting the best population, we consider a general class of selection rules based on the natural estimators of $\theta_i, i = 1, \dots, k$. Under the entropy loss function, we consider the problem of estimating the scale parameter θ_S of the population selected using a fixed selection rule from this class. We derive the uniformly minimum risk unbiased estimator of θ_S and two natural estimators of θ_S are also considered. We derive a general result for improving a scale invariant estimator of θ_S under the entropy loss function. A simulation study on the performances of various competing estimators of θ_S is also reported. Finally, we provide similar results for the problem of estimating the scale parameter of selected population when the selection goal is that of selecting the worst uniform population.

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Key words and phrases. Estimation after selection, uniform population, inadmissible estimators, entropy loss function, natural selection rule.

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The probability that *n* random points in a disk are in convex position

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Abstract. Pick n random points $x_1, \dots, x_n$ uniformly and independently in a disk and consider their convex hull C. Let $P_D^{n,m}$ be the probability that exactly m points among the x_i's are on the boundary of the convex hull of $\{x_1, \dots, x_n\}$ (so that $P_D^{n,n}$ is the probability that the x_i's are in a convex position).

In the paper, we provide a formula for $P_D^{n,m}$.

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Key words and phrases. Random convex chain, Sylvester's problem, random polygon, geometrical probability.

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G method in action: Fast exact sampling from set of permutations of order *n* according to Mallows model through Cayley metric

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Abstract. Using G method, we give a fast exact (not approximate) Markovian method for sampling from $\mathbb{S}_n$, the set of permutations of order n, according to the Mallows model through Cayley metric (a model for ranked data). This method has something in common with the cyclic Gibbs sampler and something in common with the swapping method. The number of steps of our method is equal to the number of steps of swapping method, that is, $n - 1$; moreover, both methods use the best probability distributions on sampling, the swapping method uses uniform probability distributions while our method uses almost uniform probability distributions (all the components of an almost uniform probability distribution are, here, identical, excepting at most one of them). But, besides sampling, we can do other things for the Mallows model through Cayley metric—we compute the normalizing constant and, by Uniqueness theorem, certain important probabilities.

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A bivariate optimal replacement policy with cumulative repair cost limit for a two-unit system under shock damage interaction

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Abstract. In this paper, a bivariate (n, k) replacement policy with cumulative repair cost limit for a two-unit system is studied, in which the system is subjected to shock damage interaction between units. Each unit 1 failure causes random damage to unit 2 and these damages are additive. Unit 2 will fail when the total damage of unit 2 exceed a failure level K, and such a failure makes unit 1 fail simultaneously, resulting in a total failure. When unit 1 failure occurs, if the cumulative repair cost till to this failure is less than a predetermined limit L, then unit 1 is corrected by minimal repair, otherwise, the system is preventively replaced. The system is also replaced at the nth unit 1 failure, or at damage level k $(<K)$ of unit 2, or at total failure. The explicit expression of the long-term expected cost per unit time is derived and the corresponding optimal bivariate replacement policy can be determined analytically or numerically. Finally, a numerical example is given to illustrate the theoretical results for the proposed model.

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Concentration function for the skew-normal and skew-*t* distributions, with application in robust Bayesian analysis

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Abstract. Data from many applied fields exhibit both heavy tail and skewness behavior. For this reason, in the last few decades, there has been a growing interest in exploring parametric classes of skew-symmetrical distributions. A popular approach to model departure from normality consists of modifying a symmetric probability density function in a multiplicative fashion, introducing skewness. An important issue, addressed in this paper, is the introduction of some measures of distance between skewed versions of probability densities and their symmetric baseline. Different measures provide different insights on the departure from symmetric density functions: we analyze and discuss L_1 distance, J-divergence and the concentration function in the normal and Student-t cases. Multiplicative contaminations of distributions can be also considered in a Bayesian framework as a class of priors and the notion of distance is here strongly connected with Bayesian robustness analysis: we use the concentration function to analyze departure from a symmetric baseline prior through multiplicative contamination prior distributions for the location parameter in a Gaussian model.

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Asymptotics for sparse exponential random graph models

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Abstract. We study the asymptotics for sparse exponential random graph models where the parameters may depend on the number of vertices of the graph. We obtain exact estimates for the mean and variance of the limiting probability distribution and the limiting log partition function of the edge-(single)-star model. They are in sharp contrast to the corresponding asymptotics in dense exponential random graph models. Similar analysis is done for directed sparse exponential random graph models parametrized by edges and multiple outward stars.

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