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Preface

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Numerical stability enhancements in beta autoregressive moving average model estimation

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Abstract. This paper introduces a ridge penalization scheme to enhance the numerical stability of conditional maximum likelihood estimation of the parameters indexing the βARMA model. The proposed approach involves adding a simple penalty term to the conditional log-likelihood function to enhance its curvature. This modification reduces the chance of convergence failures and implausible estimates. We also present a bootstrap-based parameter estimation strategy. It is particularly useful when penalization alone is insufficient to address numerical issues, providing a complementary solution for obtaining more reliable estimates. Our numerical results show the effectiveness of the proposed approaches in addressing numerical instability issues in βARMA parameter estimation. Two empirical applications are presented and discussed.

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Comparison of zero-inflated and hurdle INAR(1) processes for modeling count data

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Abstract. Modeling count time series is essential in many fields, especially when data exhibit complex features such as overdispersion and zero modification. While the first-order integer-valued autoregressive (INAR(1)) model is a fundamental tool, it often fails under these conditions. This study explores and systematically compares two extensions: zero-inflated INAR(1) and hurdle INAR(1), utilizing both Poisson and negative binomial innovations. We propose a unified Bayesian framework using Hamiltonian Monte Carlo in Stan to assess performance under zero-inflated and zero-deflated scenarios. Simulation results reveal a critical trade-off: in zero-inflated settings, zero-inflated models generally offer superior predictive fit, whereas hurdle models provide more accurate recovery of structural zero parameters. In zero-deflated settings, however, zero-inflated models fail structurally, making hurdle models the only viable alternative. These theoretical findings are corroborated by applications to two contrasting datasets from the same urban census tract: drug-related offenses (zero-inflated) and sex offenses (zero-deflated). To support reproducibility and broader adoption, we provide an open-source R package, `ZIHINAR1`, available on CRAN and GitHub, for model fitting and comparison. These findings offer practical guidance for selecting models that accommodate complex zero structures in discrete-valued time series.

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A Unit Gompertz ARMA model for bounded variables with time-varying quantiles

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Abstract. The Unit Gompertz (UGo) distribution has two parameters and is adequate for modeling data with support in the interval (0, 1). It was introduced as an alternative to the beta and Kumaraswamy distribution for modeling double-bounded variables. The UGo distribution can accommodate asymmetric data and has been applied in various situations, such as environmental studies, industrial applications, and survival analysis. An attractive characteristic of the UGo is its closed-form expression for the quantile function. It allows for the formulation of a quantile-based parameterization and accommodates different dependence structures for modeling the conditional quantiles. Therefore, in this study, we introduce a simple alternative for modeling double-bounded variables under serial correlation in a conditional quantile of the UGo distribution. The so-called UGo-ARMA is constructed considering an autoregressive moving average structure using the UGo distribution as the random component. The maximum likelihood method was used for parameter estimation. Subsequently, Monte Carlo simulations are conducted to investigate the performance of the maximum likelihood estimators and the asymptotic confidence intervals of the parameters. The proposed model is an alternative for modeling double-bounded variables with serial correlation, especially in contexts where UGo has already proven competitive with classic unitary distributions. To illustrate the practical relevance of the proposed model, we apply it to a financial time series: the average monthly interest rate for credit card installment operations in Brazil. The results highlight the model's ability to capture serial dependence and distributional features typically found in real-world bounded data.

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A Cramér–von Mises-type statistic for identifying jump variations in high-frequency time series

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Abstract. We propose a Cramér–von Mises-type statistic for testing the null hypothesis of no jumps in high-frequency financial series. The test is constructed from the integrated squared distance between the empirical distribution of devolatilized and truncated intraday returns and the theoretical distribution implied by a continuous no-jump diffusion model, incorporating truncation and local volatility adjustments based on established approaches. Simulation results show that the statistic achieves accurate size and strong power, outperforming the Kolmogorov–Smirnov test. An empirical application to high-frequency transaction prices of Google, Apple, and Goldman Sachs confirms that the proposed statistic successfully captures jump-driven dynamics in the series.

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Dynamic sparsity on dynamic regression models

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Abstract. In the present work, we consider variable selection and shrinkage for the Gaussian dynamic linear regression within a Bayesian framework. In particular, we propose a novel method that allows for time-varying sparsity, based on an extension of spike-and-slab priors for dynamic models. This is done by assigning appropriate Markov switching priors for the time-varying coefficients' variances, extending the previous work of (*The Annals of Statistics* **33** (2005) 730–773). Furthermore, we investigate different priors, including the common Inverted gamma prior for the process variances, and other mixture prior distributions such as Gamma priors for both the spike and the slab, which leads to a mixture of Normal-Gamma priors (*Bayesian Analysis* **5** (2010) 171–188) for the coefficients. In this sense, our prior can be viewed as a dynamic variable selection prior, which induces either smoothness (through the slab) or shrinkage towards zero (through the spike) at each time point. The MCMC method used for posterior computation uses Markov latent variables that can assume binary regimes at each time point to generate the coefficients' variances. In that way, our model is a dynamic mixture model; thus, we could use the algorithm of (*Journal of the American Statistical Association* **95** (2000) 819–828) to generate the latent processes without conditioning on the states. Finally, our approach is exemplified through simulated examples and a real data application.

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Uniform large deviation principles for SDEs under locally weak monotonicity conditions

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Abstract. The large deviation principles characterize the exponential decay rates of the probabilities of rare events. The uniform large deviation principles (ULDP) is often used to investigate the metastable behavior of random dynamical systems. In this paper, we provide a criterion on ULDP for stochastic differential equations under locally weak monotone conditions and highly generalized Lyapunov conditions, which includes not only the stochastic Hamiltonian systems and stochastic Brusselator model but also some equations with irregular coefficients. The weak convergence method plays an important role in obtaining the ULDP. This result extends the scope of applications of the main theorem in (*Bernoulli* **30** (2024) 332–345).

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Zero-adjusted defective Gompertz model with gamma frailty for survival data

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Abstract. In this article, we propose the zero-adjusted defective Gompertz model incorporating gamma frailty, aimed at jointly modeling survival data where excess zeros and cure fractions coexist, a frequent challenge in biomedical and public health studies. Traditional survival models often fail to capture these complexities simultaneously, limiting their applicability in real-world medical data. The proposed approach integrates the flexibility of the Gompertz distribution with structural adjustments for zero inflation and defective survival functions, while the inclusion of a gamma frailty term accounts for unobserved heterogeneity at the individual level. Through extensive Monte Carlo simulations and bootstrap analyses, we demonstrate the model's consistency and reliability of parameter estimates. Applications to real medical data, including insulin use among pregnant women with gestational diabetes, illustrate the model's practical utility in accurately identifying both cured and zero-adjusted subpopulations, offering a robust and versatile framework for analyzing complex survival patterns in health research.

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