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BJPS is an IMS supported journal   ISSN 0103-0752   Brazilian Statistical Association

Brazilian Journal of Probability and Statistics

Volume 40 • Number 2 • June 2026

Brazilian Journal of Probability and Statistics

Volume 40 • Number 2 • June 2026

www.imstat.org/bjps

ISSN 0103-0752 (Print) ISSN 2317-6199 (Online), Volume 40, Number 2, June 2026. Published quarterly by the Brazilian Statistical Association.

POSTMASTER:

Send address changes to Brazilian Journal of Probability and Statistics, Institute of Mathematical Statistics, Dues and Subscriptions Office, PO Box 729, Middletown, Maryland 21769, USA.

Brazilian Statistical Association members should send address changes to Rua do Matão, 1010 sala 250A, 05508-090 São Paulo/SP Brazil (address of the BSA office).

Printed in the United States of America

Partial financial support:
CNPq and CAPES (Brazil).

Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 135–161
https://doi.org/10.1214/26-BJPS656

On estimation of joint cumulative distribution of estimators in partially linear regression

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Abstract. We consider a generalized partially linear regression model of the form

$$Y_i = \mathbf{X}_i^\top \boldsymbol{\beta} + m(\mathbf{V}_i) + \Phi_i, \quad i = 1, \dots, n,$$

where $\mathbf{X}_i \in \mathbb{R}^p$ and $\mathbf{V}_i \in \mathbb{R}^q$ are mutually independent covariate vectors entering the model parametrically and nonparametrically, respectively. The term $\mathbf{X}_i^\top \boldsymbol{\beta}$ defines the linear component and $m(\mathbf{V}_i)$ is an unknown smooth function. Under the independence assumption, the joint cumulative distribution function (CDF) of the regression components is expected to factorize into the product of their marginal CDFs. We estimate the parametric and nonparametric components using standard procedures and construct a kernel-based estimator for the joint CDF of $A_i = \mathbf{X}_i^\top \widehat{\boldsymbol{\beta}}andB_i = \widehat{m}(\mathbf{V}_i)$, where the latter is obtained using the *local polynomial smoothing technique*. The proposed methodology incorporates Jacobian transformation and bandwidth adaptation to improve estimation. A Monte Carlo simulation study illustrates the performance of the estimator and supports the theoretical result that the joint distribution aligns with the product of the marginals under independence. The real data analysis confirms the practical effectiveness and robustness of the proposed semiparametric estimation framework, demonstrating its ability to capture meaningful linear effects and flexible nonlinear patterns while accurately representing joint distributional structure in applied settings.

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Das, S. (2026). Supplement to "On estimation of joint cumulative distribution of estimators in partially linear regression." https://doi.org/10.1214/26-BJPS656SUPP

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Key words and phrases. Semiparametric models, generalized partially linear regression, parametric and nonparametric components, joint distribution estimation, Kernel smoothing, local polynomial regression, bandwidth selection, Monte Carlo simulation, Jacobian transformation, cumulative distribution function.

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Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 162–185
https://doi.org/10.1214/26-BJPS657

The Bayes factor reversal paradox: A unified theory across statistical tests

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Abstract. The Bayes factor reversal (BFR) paradox shows that for any significant result under a point-null hypothesis, there exists a prior scale τ^*—the "flip point"—where $\mathrm{BF}_{01} = 1$. Below τ^*, evidence favors H_1; above, it favors H_0. We prove this phenomenon is universal across thirteen standard test families: z-tests, t-tests (one-sample, two-sample, paired), one-way ANOVA, chi-squared, proportion tests, correlation, regression, and nonparametric tests (Mann–Whitney and Wilcoxon signed-rank). The mathematical mechanism is the same in every case: continuity, a unique minimum, and divergence of $\mathrm{BF}_{01}(\tau)$ guarantee a unique flip point. We propose the flip point as a robustness diagnostic: just as power analysis quantifies frequentist fragility to sample size, the flip point quantifies Bayesian fragility to prior specification. When τ^* lies within the range of common default priors, conclusions are sensitive to arbitrary software choices. We provide closed-form solutions where available, R code, and an interactive web-based Flip Point Calculator covering all test families, enabling researchers to assess and report the robustness of their Bayesian inferences.

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Key words and phrases. Bayes factor, hypothesis testing, prior sensitivity, flip point, robustness diagnostic, ANOVA, regression, correlation, BFR paradox.

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Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 186–202
https://doi.org/10.1214/26-BJPS658

Two-dimensional index for measuring deviance from quasi-symmetry corresponding to conditional quasi-symmetry for ordinal square contingency tables

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Abstract. This study proposes a two-dimensional index for measuring the degree of deviance from the quasi-symmetry (QS) model corresponding to the conditional QS (CQS) model for ordinal square contingency tables. The CQS model indicates the structure of asymmetry for the local odds ratios. Previous study proposed a two-dimensional index for measuring the degree of deviance from the QS model corresponding to the extended QS (EQS) which has a different probability structure from the CQS model. The asymmetry parameters of the CQS and EQS models indicate both the degree and direction of deviance from the QS model. When the CQS (or EQS) model does not fit for the presented data well, we cannot evaluate them using the asymmetry parameter. The two-dimensional index can address this issue. The proposed two-dimensional index is assembled by combining two new sub-indexes. One represents the degree of deviance from the QS model corresponding to the CQS model, and the other represents its direction. Additionally, this study derives a plug-in estimator and an approximate confidence region for the proposed two-dimensional index. For comparing several datasets, we reveal that the magnitude relationship between the degrees of deviance from the QS model changes by whether the proposed or existing two-dimensional index is used.

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Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 203–234
https://doi.org/10.1214/26-BJPS659

On general weighted extropy of percentile ranked set sampling

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Abstract. Extropy, introduced as a complementary measure to Shannon entropy, has recently received considerable attention in probability and statistics due to its ability to capture uncertainty from a perspective different from entropy. Several weighted and generalized versions of extropy have been proposed to incorporate the magnitude of the underlying random variable. In parallel, ranked set sampling and its variants have emerged as efficient alternatives to simple random sampling when ranking is inexpensive relative to exact measurement. In this paper, we develop a comprehensive study of the *general weighted extropy* (GWE) under the framework of Percentile Ranked Set Sampling (PRSS). Explicit expressions for the GWE of PRSS data are derived in terms of density–quantile functions, allowing tractable analysis. We investigate fundamental properties of the proposed measure, including monotonicity, stochastic ordering results, and bounds, and establish comparisons with the corresponding GWE under simple random sampling. The behavior of cumulative past extropy under PRSS is also examined. Furthermore, several characterization results are obtained, including characterizations of symmetric distributions and the exponential distribution based on the GWE of PRSS data. A Monte Carlo simulation study is conducted to compare the performance of extropy estimators based on PRSS, extreme ranked set sampling, and simple random sampling. The results demonstrate that PRSS-based estimators provide noticeable efficiency gains over their competitors in terms of mean squared error. Overall, the findings highlight the advantages of PRSS for uncertainty measurement and extend the theory of generalized weighted extropy to a flexible and practically relevant sampling scheme.

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Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 235–257
https://doi.org/10.1214/26-BJPS662

Double random environment binomial thinning INAR(1) process with Poisson or geometric marginal

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Abstract. The manuscript introduces a new first-order integer-valued autoregressive process in a random environment, based on the binomial thinning operator. The proposed model is driven by two control processes: the first determines the marginal distribution, while the second governs the correlation structure and the dynamic behavior of the series. The inclusion of two control processes enhances the model's flexibility and allows for a more accurate fit to the data. Two variants of the model are considered—one with a Poisson marginal distribution and the other with a geometric marginal distribution—and both specifications are theoretically analyzed. The fundamental properties of the model are investigated in detail, providing comprehensive theoretical insight into its structure, moment characteristics, and the dependence structure it generates. For the estimation of unknown parameters, two methods are employed: the Yule–Walker estimator and the conditional maximum likelihood estimator. Their performance is evaluated through extensive simulation studies. Finally, the practical applicability of the proposed model is illustrated using real-world data.

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Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 258–276
https://doi.org/10.1214/26-BJPS663

The Beta-pG family of distributions for cure rate regression models

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Abstract. In this paper, we introduce the Beta-pG family of distributions as a flexible framework for modeling long-term survival data. This family extends the traditional Beta-G distribution by incorporating a new parameter p, from which a nonzero long-term survivor fraction is induced through the limiting survival probability. This structure allows the model to capture survival patterns in which a portion of the population remains unaffected by the event of interest over extended follow-up. In particular, we focus on the case where G, the baseline distribution, is specified as the exponential distribution, leading to the Beta-pExponential model. Parameter estimation is performed using maximum likelihood method, allowing likelihood-based inference for the model parameters and the long-term survivor proportions. Monte Carlo simulation studies are conducted to evaluate the finite-sample performance of the estimators across various scenarios. The proposed model is then applied to two melanoma survival datasets, including data from a clinical trial and a population-based cancer registry. In both applications, the fitted model captures the main features of the observed survival curves and provides direct estimates of the long-term survivor proportions. These results illustrate the usefulness of the Beta-pG family as an alternative framework for analyzing survival data with long-term survivors.

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Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 277–307
https://doi.org/10.1214/26-BJPS664

A goodness-of-fit test for the geometric maximum compound logistic distribution model

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Abstract. Based on independent copies of a bivariate random vector (M, N), with positive integer-valued component N, we consider testing the composite null hypothesis that (M, N) follows a geometric maximum compound logistic distribution. This distributional model is of interest for example in hydrology, where N models the number of floods, and M the maximum flood water level during a certain time period. The geometric maximum compound logistic distribution is characterized in the sense that a special transform of (M, N) fulfills a specific equation. We suggest a weighted integral of an expression obtained by replacing the function part of this equation by an empirical counterpart as test statistic, propose a parametric bootstrap procedure to get critical values, give a thorough discussion that the procedure works, and present a simulation study on its performance. The test is applied to a hydrological data set. A new goodness-of-fit test for the logistic distribution is obtained as a special case of the novel approach.

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Key words and phrases. Bootstrapping, empirical processes, geometric maximum compound logistic distribution, goodness-of-fit.

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Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 308–333
https://doi.org/10.1214/26-BJPS665

Bayesian variable selection for function-on-scalar regression models: A comparative analysis

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Abstract. In this work, we developed a new Bayesian method for variable selection in function-on-scalar regression (FOSR). Our method uses a hierarchical Bayesian structure and latent variables to enable an adaptive covariate selection process for FOSR. Extensive simulation studies show the proposed method's main properties, such as its accuracy in estimating the coefficients and its high capacity to select variables correctly. Furthermore, we conducted a substantial comparative analysis with the main competing methods: the BGLSS (Bayesian Group Lasso with Spike and Slab prior) method, the group LASSO (Least Absolute Shrinkage and Selection Operator), the group MCP (Minimax Concave Penalty), and the group SCAD (Smoothly Clipped Absolute Deviation). Our results demonstrate that the proposed methodology is superior in correctly selecting covariates compared with the existing competing methods while maintaining an excellent level of goodness of fit. In contrast, the competing methods could not balance selection accuracy with goodness of fit. We also considered a COVID-19 dataset and some socioeconomic data from Brazil as an application and obtained sound results. In short, the proposed Bayesian variable selection model is highly competitive, showing significant predictive and selective quality.

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Key words and phrases. Bayesian inference, functional data, functional data analysis, variable selection, latent variable, function-on-scalar regression.

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Brazilian Journal of Probability and Statistics
2026, Vol. 40, No. 2, 334–348
https://doi.org/10.1214/26-BJPS666

A copula-based joint regression analysis including Granger causality

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Abstract. Granger causality models are widely used in time series analysis to manage causal relationships between variables based on temporal precedence. Traditional Granger causality models always assume linear relationships and normality assumptions, which often fail to capture the complexities of real-world data involving nonlinear and non-normal behavior. In this work, we propose a copula-based approach that offers a novel alternative method by capturing nonlinearity and non-normality in multivariate time series modeling. We develop a copula-based joint regression model tailored for Granger causality analysis and carry out some predictions in these models. We assess and validate the performance of our method using simulation studies in different scenarios. Two applications of our proposal in real-world data analysis are also presented. Our findings highlight the superiority of copula-based approaches compared to other Granger causality models in complex time series data.

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