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Sufficientness Postulates for Gibbs-Type Priors and Hierarchical Generalizations

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Abstract. A fundamental problem in Bayesian nonparametrics consists of selecting a prior distribution by assuming that the corresponding predictive probabilities obey certain properties. An early discussion of such a problem, although in a parametric framework, dates back to the seminal work by English philosopher W. E. Johnson, who introduced a noteworthy characterization for the predictive probabilities of the symmetric Dirichlet prior distribution. This is typically referred to as Johnson's "sufficientness" postulate. In this paper, we review some nonparametric generalizations of Johnson's postulate for a class of nonparametric priors known as species sampling models. In particular, we revisit and discuss the "sufficientness" postulate for the two parameter Poisson–Dirichlet prior within the more general framework of Gibbs-type priors and their hierarchical generalizations.

Key words and phrases: Bayesian nonparametrics, Dirichlet and two parameter Poisson–Dirichlet process, discovery probability, Gibbs-type species sampling models, hierarchical species sampling models, Johnson's "sufficientness" postulate, Pólya-like urn scheme, predictive probabilities.

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Spherical Process Models for Global Spatial Statistics

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Abstract. Statistical models used in geophysical, environmental, and climate science applications must reflect the curvature of the spatial domain in global data. Over the past few decades, statisticians have developed covariance models that capture the spatial and temporal behavior of these global data sets. Though the geodesic distance is the most natural metric for measuring distance on the surface of a sphere, mathematical limitations have compelled statisticians to use the chordal distance to compute the covariance matrix in many applications instead, which may cause physically unrealistic distortions. Therefore, covariance functions directly defined on a sphere using the geodesic distance are needed. We discuss the issues that arise when dealing with spherical data sets on a global scale and provide references to recent literature. We review the current approaches to building process models on spheres, including the differential operator, the stochastic partial differential equation, the kernel convolution, and the deformation approaches. We illustrate realizations obtained from Gaussian processes with different covariance structures and the use of isotropic and nonstationary covariance models through deformations and geographical indicators for global surface temperature data. To assess the suitability of each method, we compare their log-likelihood values and prediction scores, and we end with a discussion of related research problems.

Key words and phrases: Axial symmetry, chordal distance, geodesic distance, nonstationarity, smoothness, sphere.

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The General Structure of Evidence Factors in Observational Studies

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Abstract. The general structure of evidence factors is examined in terms of the knit product of two permutation groups. An observational or nonrandomized study of treatment effects has two evidence factors if it permits two (nearly) independent tests of the null hypothesis of no treatment effect and two (nearly) independent sensitivity analyses for those tests. Either of the two tests may be biased by nonrandom treatment assignment, but certain biases that would invalidate one test would have no impact on the other, so if the two tests concur, then some aspects of biased treatment assignment have been partially addressed. Expressed in terms of the knit product of two permutation groups, the structure of evidence factors is simpler and less cluttered, but at the same time more general and easier to apply in a new context. The issues are exemplified by an observational study of cigarette smoking as a cause of periodontal disease.

Key words and phrases: Evidence factor, knit product, permutation group, permutation inference, randomization inference, semidirect product, sensitivity analysis, wreath product, Zappa–Szep product.

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Hierarchical Sparse Modeling: A Choice of Two Group Lasso Formulations

Xiaohan Yan and Jacob Bien

Abstract. Demanding sparsity in estimated models has become a routine practice in statistics. In many situations, we wish to require that the sparsity patterns attained honor certain problem-specific constraints. *Hierarchical sparse modeling* (HSM) refers to situations in which these constraints specify that one set of parameters be set to zero whenever another is set to zero. In recent years, numerous papers have developed convex regularizers for this form of sparsity structure, which arises in many areas of statistics including interaction modeling, time series analysis, and covariance estimation. In this paper, we observe that these methods fall into two frameworks, the *group lasso* (GL) and *latent overlapping group lasso* (LOG), which have not been systematically compared in the context of HSM. The purpose of this paper is to provide a side-by-side comparison of these two frameworks for HSM in terms of their statistical properties and computational efficiency. We call special attention to GL's more aggressive shrinkage of parameters deep in the hierarchy, a property not shared by LOG. In terms of computation, we introduce a finite-step algorithm that exactly solves the proximal operator of LOG for a certain simple HSM structure; we later exploit this to develop a novel path-based block coordinate descent scheme for general HSM structures. Both algorithms greatly improve the computational performance of LOG. Finally, we compare the two methods in the context of covariance estimation, where we introduce a new sparsely-banded estimator using LOG, which we show achieves the statistical advantages of an existing GL-based method but is simpler to express and more efficient to compute.

Key words and phrases: Hierarchical sparsity, convex regularization, group lasso, latent overlapping group lasso.

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Instrumental Variable Estimation with a Stochastic Monotonicity Assumption

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Abstract. The instrumental variables (IV) method provides a way to estimate the causal effect of a treatment when there are unmeasured confounding variables. The method requires a valid IV, a variable that is independent of the unmeasured confounding variables and is associated with the treatment but which has no effect on the outcome beyond its effect on the treatment. An additional assumption often made is deterministic monotonicity, which says that for each subject, the level of the treatment that a subject would take is a monotonic increasing function of the level of the IV. However, deterministic monotonicity is sometimes not realistic. We introduce a stochastic monotonicity assumption, a relaxation that only requires a monotonic increasing relationship to hold across subjects between the IV and the treatments conditionally on a set of (possibly unmeasured) covariates. We show that under stochastic monotonicity, the IV method identifies a weighted average of treatment effects with greater weight on subgroups of subjects on whom the IV has a stronger effect. We provide bounds on the global average treatment effect under stochastic monotonicity and a sensitivity analysis for violations of stochastic monotonicity. We apply the methods to a study of the effect of premature babies being delivered in a high technology neonatal intensive care unit (NICU) vs. a low technology unit.

Key words and phrases: Causal inference, observational study, instrumental variable, two stage least squares.

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The Coordinate-Based Meta-Analysis of Neuroimaging Data

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Abstract. Neuroimaging meta-analysis is an area of growing interest in statistics. The special characteristics of neuroimaging data render classical meta-analysis methods inapplicable and therefore new methods have been developed. We review existing methodologies, explaining the benefits and drawbacks of each. A demonstration on a real dataset of emotion studies is included. We discuss some still-open problems in the field to highlight the need for future research.

Key words and phrases: Meta-analysis, neuroimaging, functional magnetic resonance imaging.

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Contemporary Frequentist Views of the 2 x 2 Binomial Trial

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Abstract. The 2×2 table is the simplest of data structures yet it is of immense practical importance. It is also just complex enough to provide a theoretical testing ground for general frequentist methods. Yet after 70 years of debate, its correct analysis is still not settled. Rather than recount the entire history, our review is motivated by contemporary developments in likelihood and testing theory as well as computational advances. We will look at both conditional and unconditional tests. Within the conditional framework, we explain the relationship of Fisher's test with variants such as mid-p and Liebermeister's test, as well as modern developments in likelihood theory, such as p^* and approximate conditioning. Within an unconditional framework, we consider four modern methods of correcting approximate tests to properly control size by accounting for the unknown value of the nuisance parameter: maximisation (M), partial maximisation (B), estimation (E) and estimation followed by maximisation (E + M). Under the conditional model, we recommend Fisher's test. For the unconditional model, amongst standard approximate methods, Liebermeister's tests come closest to controlling size. However, our best recommendation is the E procedure applied to the signed root likelihood statistic, as this performs very well in terms of size and power and is easily computed. We support our assertions with a numerical study.

Key words and phrases: Approximate conditioning, binomial trial, conditional test, exact tests, Fisher test, Liebermeister test, mid-p test, parametric bootstrap, unconditional test.

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Elo Ratings and the Sports Model: A Neglected Topic in Applied Probability?

David Aldous

Abstract. In a simple model for sports, the probability A beats B is a specified function of their difference in strength. One might think this would be a staple topic in Applied Probability textbooks (like the Galton–Watson branching process model, for instance) but it is curiously absent. Our first purpose is to point out that the model suggests a wide range of questions, suitable for "undergraduate research" via simulation but also challenging as professional research. Our second, more specific, purpose concerns Elo-type rating algorithms for tracking changing strengths. There has been little foundational research on their accuracy, despite a much-copied "30 matches suffice" claim, which our simulation study casts doubt upon.

Key words and phrases: Elo rating, Bradley–Terry model, dynamic ratings, sports forecasting.

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On a General Definition of Depth for Functional Data

Irène Gijbels and Stanislav Nagy

Abstract. In this paper, we provide an elaboration on the desirable properties of statistical depths for functional data. Although a formal definition has been put forward in the literature, there are still several unclarities to be tackled, and further insights to be gained. Herein, a few interesting connections between the wanted properties are found. In particular, it is demonstrated that the conditions needed for some desirable properties to hold are extremely demanding, and virtually impossible to be met for common depths. We establish adaptations of these properties which prove to be still sensible, and more easily met by common functional depths.

Key words and phrases: Data depth, functional data, multivariate statistics, robustness.

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Correction to "A Topologically Valid Definition of Depth for Functional Data"

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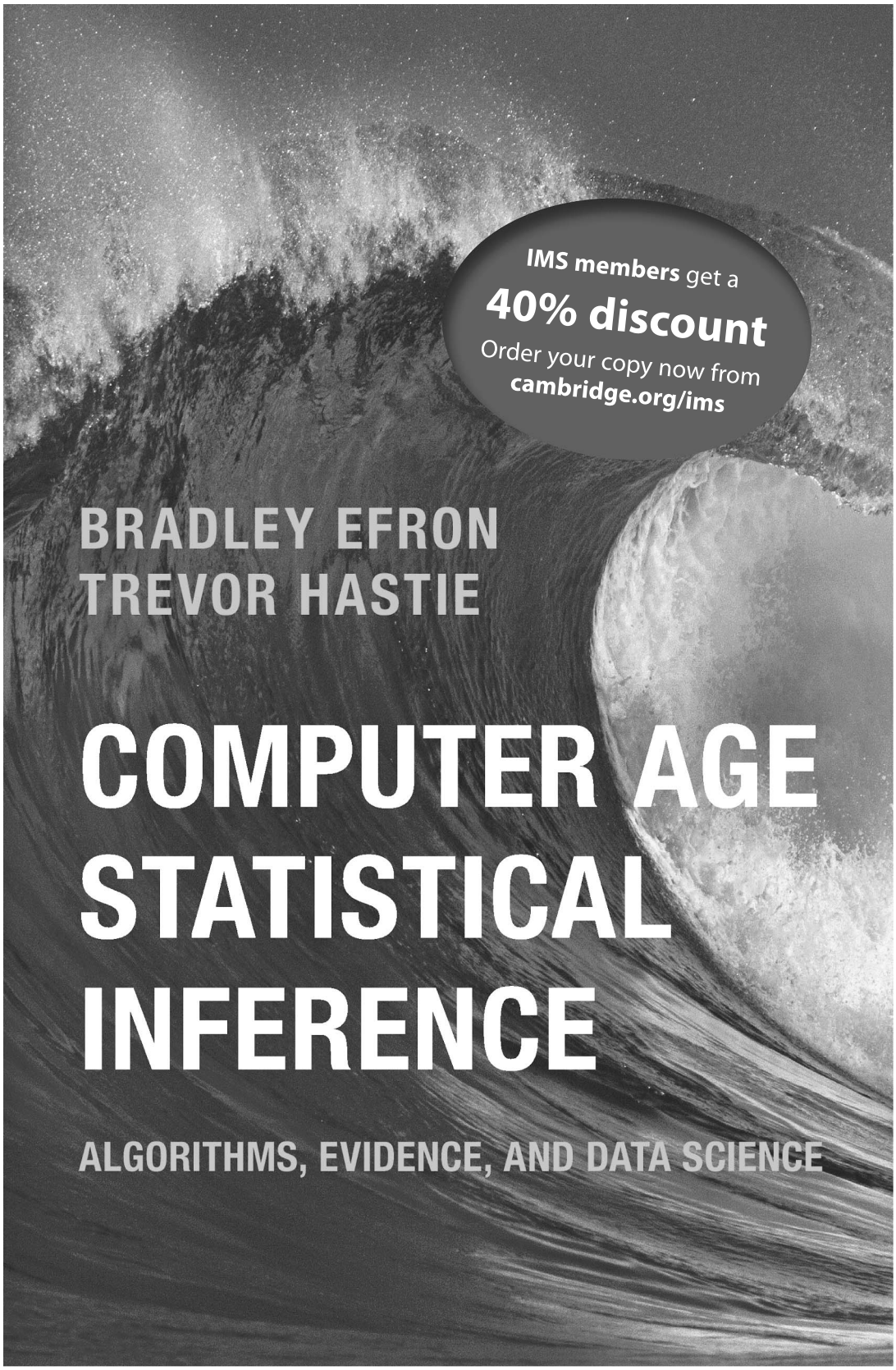
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