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ROS Regression: Integrating Regularization with Optimal Scaling Regression

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Abstract. We present a methodology for multiple regression analysis that deals with categorical variables (possibly mixed with continuous ones), in combination with regularization, variable selection and high-dimensional data ($P \gg N$). Regularization and optimal scaling (OS) are two important extensions of ordinary least squares regression (OLS) that will be combined in this paper. There are two data analytic situations for which optimal scaling was developed. One is the analysis of categorical data, and the other the need for transformations because of nonlinear relationships between predictors and outcome. Optimal scaling of categorical data finds quantifications for the categories, both for the predictors and for the outcome variables, that are optimal for the regression model in the sense that they maximize the multiple correlation. When nonlinear relationships exist, nonlinear transformation of predictors and outcome maximize the multiple correlation in the same way. We will consider a variety of transformation types; typically we use step functions for categorical variables, and smooth (spline) functions for continuous variables. Both types of functions can be restricted to be monotonic, preserving the ordinal information in the data. In combination with optimal scaling, three popular regularization methods will be considered: Ridge regression, the Lasso and the Elastic Net. The resulting method will be called ROS Regression (Regularized Optimal Scaling Regression). The OS algorithm provides straightforward and efficient estimation of the regularized regression coefficients, automatically gives the Group Lasso and Blockwise Sparse Regression, and extends them by the possibility to maintain ordinal properties in the data. Extended examples are provided.

Key words and phrases: Lasso and Elastic Net regularization for nominal and ordinal data, monotonic group Lasso, regularization for categorical high-dimensional data, optimal scaling, linearization of nonlinear relationships, monotonic step functions and splines.

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An Overview of Semiparametric Extensions of Finite Mixture Models

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Abstract. Finite mixture models have offered a very important tool for exploring complex data structures in many scientific areas, such as economics, epidemiology and finance. Semiparametric mixture models, which were introduced into traditional finite mixture models in the past decade, have brought forth exciting developments in their methodologies, theories, and applications. In this article, we not only provide a selective overview of the newly-developed semiparametric mixture models, but also discuss their estimation methodologies, theoretical properties if applicable, and some open questions. Recent developments are also discussed.

Key words and phrases: EM algorithm, mixture models, mixture regression models, semiparametric mixture models.

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Lasso Meets Horseshoe: A Survey

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Abstract. The goal of this paper is to contrast and survey the major advances in two of the most commonly used high-dimensional techniques, namely, the Lasso and horseshoe regularization. Lasso is a gold standard for predictor selection while horseshoe is a state-of-the-art Bayesian estimator for sparse signals. Lasso is fast and scalable and uses convex optimization whilst the horseshoe is nonconvex. Our novel perspective focuses on three aspects: (i) theoretical optimality in high-dimensional inference for the Gaussian sparse model and beyond, (ii) efficiency and scalability of computation and (iii) methodological development and performance.

Key words and phrases: Global-local priors, horseshoe, horseshoe+, hyperparameter tuning, Lasso, regression, regularization, sparsity.

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The Geometry of Continuous Latent Space Models for Network Data

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Abstract. We review the class of continuous latent space (statistical) models for network data, paying particular attention to the role of the geometry of the latent space. In these models, the presence/absence of network dyadic ties are assumed to be conditionally independent given the dyads' unobserved positions in a latent space. In this way, these models provide a probabilistic framework for embedding network nodes in a continuous space equipped with a geometry that facilitates the description of dependence between random dyadic ties. Specifically, these models naturally capture homophilous tendencies and triadic clustering, among other common properties of observed networks. In addition to reviewing the literature on continuous latent space models from a geometric perspective, we highlight the important role the geometry of the latent space plays on properties of networks arising from these models via intuition and simulation. Finally, we discuss results from spectral graph theory that allow us to explore the role of the geometry of the latent space, independent of network size. We conclude with conjectures about how these results might be used to infer the appropriate latent space geometry from observed networks.

Key words and phrases: Geometric curvature, graph Laplacian, latent variable, network model.

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User-Friendly Covariance Estimation for Heavy-Tailed Distributions

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Abstract. We provide a survey of recent results on covariance estimation for heavy-tailed distributions. By unifying ideas scattered in the literature, we propose user-friendly methods that facilitate practical implementation. Specifically, we introduce elementwise and spectrumwise truncation operators, as well as their M-estimator counterparts, to robustify the sample covariance matrix. Different from the classical notion of robustness that is characterized by the breakdown property, we focus on the tail robustness which is evidenced by the connection between nonasymptotic deviation and confidence level. The key insight is that estimators should adapt to the sample size, dimensionality and noise level to achieve optimal tradeoff between bias and robustness. Furthermore, to facilitate practical implementation, we propose data-driven procedures that automatically calibrate the tuning parameters. We demonstrate their applications to a series of structured models in high dimensions, including the bandable and low-rank covariance matrices and sparse precision matrices. Numerical studies lend strong support to the proposed methods.

Key words and phrases: Covariance estimation, heavy-tailed data, M-estimation, nonasymptotics, tail robustness, truncation.

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Conditionally Conjugate Mean-Field Variational Bayes for Logistic Models

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Abstract. Variational Bayes (VB) is a common strategy for approximate Bayesian inference, but simple methods are only available for specific classes of models including, in particular, representations having conditionally conjugate constructions within an exponential family. Models with logit components are an apparently notable exception to this class, due to the absence of conjugacy among the logistic likelihood and the Gaussian priors for the coefficients in the linear predictor. To facilitate approximate inference within this widely used class of models, Jaakkola and Jordan (*Stat. Comput.* **10** (2000) 25–37) proposed a simple variational approach which relies on a family of tangent quadratic lower bounds of the logistic log-likelihood, thus restoring conjugacy between these approximate bounds and the Gaussian priors. This strategy is still implemented successfully, but few attempts have been made to formally understand the reasons underlying its excellent performance. Following a review on VB for logistic models, we cover this gap by providing a formal connection between the above bound and a recent Pólya-gamma data augmentation for logistic regression. Such a result places the computational methods associated with the aforementioned bounds within the framework of variational inference for conditionally conjugate exponential family models, thereby allowing recent advances for this class to be inherited also by the methods relying on Jaakkola and Jordan (*Stat. Comput.* **10** (2000) 25–37).

Key words and phrases: EM, logistic regression, Pólya-gamma data augmentation, quadratic approximation, variational Bayes.

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Assessing the Causal Effect of Binary Interventions from Observational Panel Data with Few Treated Units

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Abstract. Researchers are often challenged with assessing the impact of an intervention on an outcome of interest in situations where the intervention is nonrandomised, the intervention is only applied to one or few units, the intervention is binary, and outcome measurements are available at multiple time points. In this paper, we review existing methods for causal inference in these situations. We detail the assumptions underlying each method, emphasize connections between the different approaches and provide guidelines regarding their practical implementation. Several open problems are identified thus highlighting the need for future research.

Key words and phrases: Causal impact, causal inference, difference-in-differences, intervention evaluation, latent factor models, panel data, synthetic controls.

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A Conversation with Peter Diggle

Peter M. Atkinson and Jorge Mateu

Abstract. Peter John Diggle was born on February 24, 1950, in Lancashire, England. Peter went to school in Scotland, and it was at the end of his school years that he found that he was good at maths and actually enjoyed it. Peter went to Edinburgh to do a maths degree, but transferred halfway through to Liverpool where he completed his degree. Peter studied for a year at Oxford and was then appointed in 1974 as a lecturer in statistics at the University of Newcastle-upon-Tyne where he gained his PhD, and was promoted to Reader in 1983. A sabbatical at the Swedish Royal College of Forestry gave him his first exposure to real scientific data and problems, prompting a move to CSIRO, Australia. After five years with CSIRO where he was Senior, then Principal, then Chief Research Scientist and Chief of the Division of Mathematics and Statistics, he returned to the UK in 1988, to a Chair at Lancaster University. Since 2011 Peter has held appointments at Lancaster and Liverpool, together with honorary appointments at Johns Hopkins, Columbia and Yale. At Lancaster, Peter was the founder and Director of the Medical Statistics Unit (1995–2001), University Dean for Research (1998–2001), EPSRC Senior Fellow (2004–2008), Associate Dean for Research at the School of Health and Medicine (2007–2011), Distinguished University Professor, and leader of the CHICAS Research Group (2007–2017). A Fellow of the Royal Statistical Society since 1974, he was a Member of Council (1983–1985), Joint Editor of *JRSSB* (1984–1987), Honorary Secretary (1990–1996), awarded the Guy Medal in Silver (1997) and the Barnett Award (2018), Associate Editor of *Applied Statistics* (1998–2000), Chair of the Research Section Committee (1998–2000), and President (2014–2016). Away from work, Peter enjoys music, playing folk-blues guitar and tenor recorder, and listening to jazz. His running days are behind him, but he can just about hold his own in mixed-doubles badminton with his family. His boyhoood hero was Stirling Moss, and he retains an enthusiasm for classic cars, not least his 1988 Porsche 924S. His favorite authors are George Orwell, Primo Levi and Nigel Slater. This interview was done prior to the fourth Spatial Statistics conference held in Lancaster, July 2017 where a session was dedicated to Peter celebrating his contributions to statistics.

Key words and phrases: CSIRO, geostatistics, Lancaster, longitudinal data analysis, point pattern analysis, spatial statistics, CHICAS.

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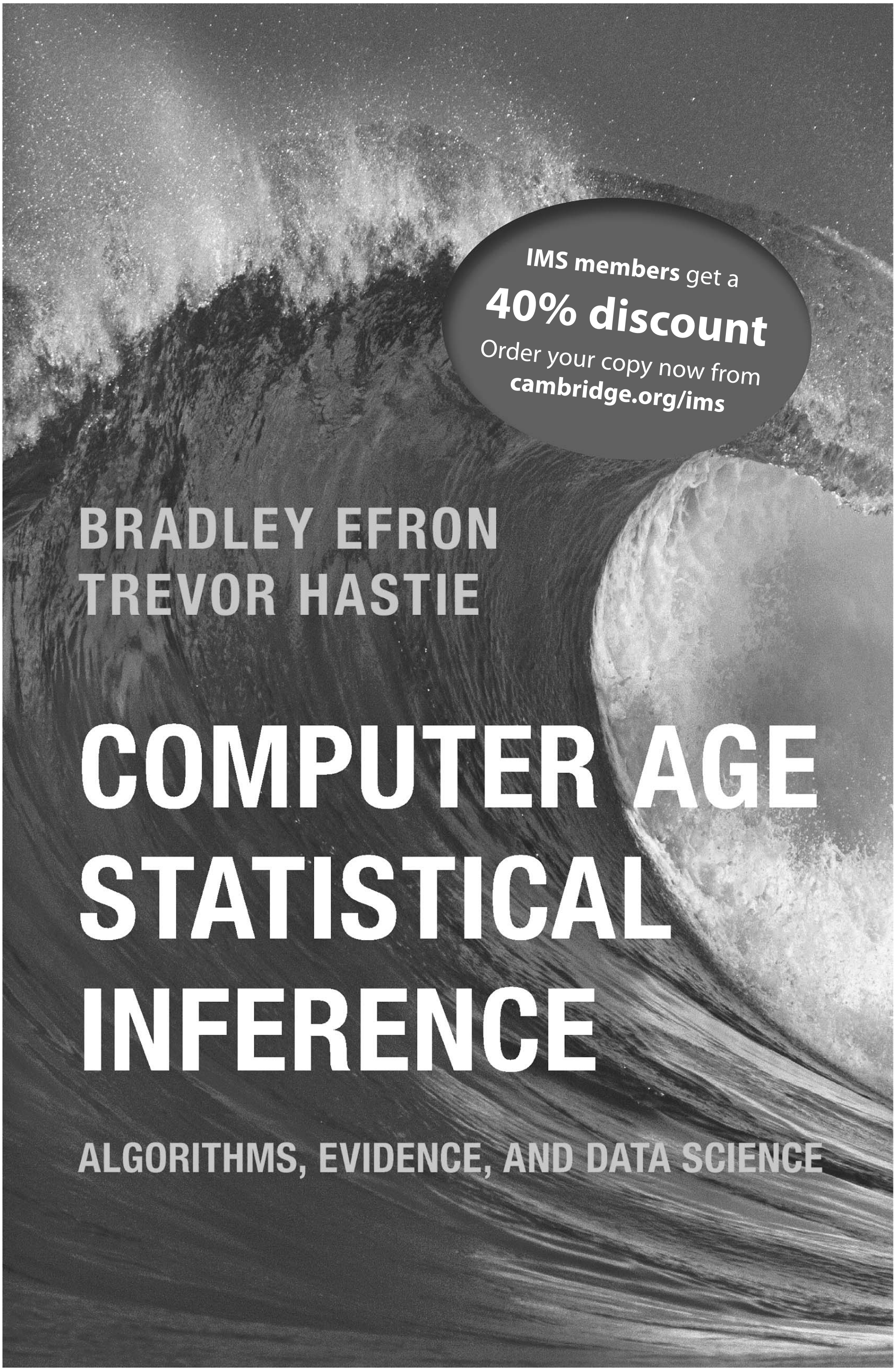
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