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Laplace's Theories of Cognitive Illusions, Heuristics and Biases

Joshua B. Miller and Andrew Gelman

Abstract. In his book from the early 1800s, *Essai Philosophique sur les Probabilités*, the mathematician Pierre-Simon de Laplace anticipated many ideas developed within the past 50 years in cognitive psychology and behavioral economics, explaining human tendencies to deviate from norms of rationality in the presence of probability and uncertainty. A look at Laplace's theories and reasoning is striking, both in how modern they seem, how much progress he made without the benefit of systematic experimentation, and the novelty of a few of his unexplored conjectures. We argue that this work points to these theories being more fundamental and less contingent on recent experimental findings than we might have thought.

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Comment: Laplace and Cognitive Illusions

Daniel Kahneman and Maya Bar-Hillel

Abstract. Reports in the 1970s of cognitive illusions in judgments of uncertainty had been anticipated by Laplace 150 years earlier. We discuss Miller and Gelman's remark that Laplace's anticipation of the main ideas of the heuristics and biases approach "gives us a new perspective on these ideas as more universal and less contingent on particular developments [that came much] later."

Key words and phrases: Cognitive illusions, heuristics and biases, judgment under uncertainty.

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Comment: Illusions, Then and Now

Glenn Shafer

Abstract. The situation of mathematical statistics today resembles the situation of Laplacean probability in the mid-19th century. Strong claims are made about how probability theory should be used. Yet it is widely misused and increasingly not used in the analysis of data. Perhaps we need a philosophy of probability that makes more modest and realistic claims.

Key words and phrases: Laplace, Cournot, cognitive illusions, norms, foundations of probability.

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Rejoinder: Laplace's theories of cognitive illusions, heuristics and biases

Joshua B. Miller and Andrew Gelman

Abstract. We appreciate the thoughtful comments from Glenn Shafer and from Daniel Kahneman and Maya Bar-Hillel and respond to each in turn.

Key words and phrases: Cognitive illusions, history of statistics, probability, psychology.

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Fano's Inequality for Random Variables

Sébastien Gerchinovitz, Pierre Ménard and Gilles Stoltz

Abstract. We extend Fano's inequality, which controls the average probability of events in terms of the average of some f-divergences, to work with arbitrary events (not necessarily forming a partition) and even with arbitrary $[0, 1]$-valued random variables, possibly in continuously infinite number. We provide two applications of these extensions, in which the consideration of random variables is particularly handy: we offer new and elegant proofs for existing lower bounds, on Bayesian posterior concentration (minimax or distribution-dependent) rates and on the regret in nonstochastic sequential learning.

Key words and phrases: Multiple-hypotheses testing, lower bounds, information theory, Bayesian posterior concentration.

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Equitability, Interval Estimation, and Statistical Power

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Abstract. Emerging high-dimensional data sets often contain many nontrivial relationships, and, at modern sample sizes, screening these using an independence test can sometimes yield too many relationships to be a useful exploratory approach. We propose a framework to address this limitation centered around a property of measures of dependence called *equitability*. Given some measure of relationship strength, an equitable measure of dependence is one that assigns similar scores to equally strong relationships of different types. We formalize equitability within a semiparametric inferential framework in terms of interval estimates of relationship strength, and we then use the correspondence of these interval estimates to hypothesis tests to show that equitability is equivalent under moderate assumptions to requiring that a measure of dependence yield well-powered tests not only for distinguishing nontrivial relationships from trivial ones but also for distinguishing stronger relationships from weaker ones. We then show that equitability, to the extent it is achieved, implies that a statistic will be well powered to detect all relationships of a certain minimal strength, across different relationship types in a family. Thus, equitability is a strengthening of power against independence that enables exploration of data sets with a small number of strong, interesting relationships and a large number of weaker, less interesting ones.

Key words and phrases: Equitability, measure of dependence, statistical power, independence test, semiparametric inference.

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LGM Split Sampler: An Efficient MCMC Sampling Scheme for Latent Gaussian Models

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Abstract. A general and flexible class of latent Gaussian models is proposed in this paper. The latent Gaussian model is adapted to the generalized additive model for location, scale and shape (GAMLSS), that is, the data density function of each data point can depend on more than a single linear predictor of the latent parameters. We refer to this framework as extended latent Gaussian models. The most commonly applied latent Gaussian models (LGMs) are such that a linear predictor is proposed only for the location parameter. Extended LGMs allow proposing linear predictors also for the scale parameter and potentially other parameters. We propose a novel computationally efficient Markov chain Monte Carlo sampling scheme for the extended LGMs which we refer to as the LGM split sampler. It is a two block Gibbs sampling scheme designed to exploit the model structure of the extended LGMs. An extended LGM is constructed for a simulated dataset and the LGM split sampler is implemented for posterior simulations. The results demonstrate the flexibility of the extended LGM framework and the efficiency of the LGM split sampler.

Key words and phrases: Bayesian hierarchical models, Gibbs sampling, latent Gaussian models, Markov chain Monte Carlo, posterior simulation.

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Checking for Prior-Data Conflict Using Prior-to-Posterior Divergences

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Abstract. When using complex Bayesian models to combine information, checking consistency of the information contributed by different components of the model for inference is good statistical practice. Here a new method is developed for detecting prior-data conflicts in Bayesian models based on comparing the observed value of a prior-to-posterior divergence to its distribution under the prior predictive distribution for the data. The divergence measure used in our model check is a measure of how much beliefs have changed from prior to posterior, and can be thought of as a measure of the overall size of a relative belief function. It is shown that the proposed method is intuitive, has desirable properties, can be extended to hierarchical settings, and is related asymptotically to Jeffreys' and reference prior distributions. In the case where calculations are difficult, the use of variational approximations as a way of relieving the computational burden is suggested. The methods are compared in a number of examples with an alternative but closely related approach in the literature based on the prior predictive distribution of a minimal sufficient statistic.

Key words and phrases: Bayesian inference, model checking, prior data-conflict, variational Bayes, Bayesian inference.

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A Generalized Approach to Power Analysis for Local Average Treatment Effects

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Abstract. This study introduces a new approach to power analysis in the context of estimating a local average treatment effect (LATE), where the study subjects exhibit noncompliance with treatment assignment. As a result of distributional complications in the LATE context, compared to the simple ATE context, there is currently no standard method of power analysis for the LATE. Moreover, existing methods and commonly used substitutes—which include instrumental variable (IV), intent-to-treat (ITT) and scaled ATE power analyses—require specifying generally unknown variance terms and/or rely upon strong and unrealistic assumptions, thus providing unreliable guidance on the power of tests of the LATE. This study develops a new approach that uses standardized effect sizes to place bounds on the power for the most commonly used estimator of the LATE, the Wald IV estimator, whereby variance terms and distributional parameters need not be specified nor assumed. Instead, in addition to the effect size, sample size and error tolerance parameters, the only other parameter that must be specified by the researcher is the compliance rate. Additional conditions can also be introduced to further narrow the bounds on the power calculation. The result is a generalized approach to power analysis in the LATE context that is simple to implement.

Key words and phrases: Experimental design, local average treatment effects, noncompliance, principal stratification, statistical power.

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On the Probability That Two Random Integers Are Coprime

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Abstract. We show that there is a nonempty class of finitely additive probabilities on $\mathbb{N}^2$ such that for each member of the class, each set with limiting relative frequency p has probability p. Hence, in that context the probability that two random integers are coprime is $6/\pi^2$. We also show that two other interpretations of "random integer," namely residue classes and shift invariance, support any number in $[0, 6/\pi^2]$ for that probability. Finally, we specify a countably additive probability space that also supports $6/\pi^2$.

Key words and phrases: Number theory, coprime, uniform distribution, finitely additive probability.

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Comparative Study of Differentially Private Data Synthesis Methods

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Abstract. When sharing data among researchers or releasing data for public use, there is a risk of exposing sensitive information of individuals in the data set. Data synthesis is a statistical disclosure limitation technique for releasing synthetic data sets with pseudo individual records. Traditional data synthesis techniques often rely on strong assumptions of a data intruder's behaviors and background knowledge to assess disclosure risk. Differential privacy (DP) formulates a theoretical approach for a strong and robust privacy guarantee in data release without having to model intruders' behaviors. Efforts have been made aiming to incorporate the DP concept in the data synthesis process. In this paper, we examine current DIfferentially Private Data Synthesis (DIPS) techniques for releasing individual-level surrogate data for the original data, compare the techniques conceptually and evaluate the statistical utility and inferential properties of the synthetic data via each DIPS technique through extensive simulation studies. Our work sheds light on the practical feasibility and utility of the various DIPS approaches, and suggests future research directions for DIPS.

Key words and phrases: Differential privacy, DIPS, sufficient statistics, parametric DIPS, nonparametric DIPS, statistical disclosure limitation.

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A Conversation with Grace Wahba

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Abstract. Grace Wahba (née Goldsmith, born August 3, 1934), I. J. Schoenberg-Hilldale Professor of Statistics at the University of Wisconsin-Madison (Emerita), is a pioneer in methods for smoothing noisy data. Her research combines theoretical analysis, computation and methodology motivated by innovative scientific applications. Best known for the development of generalized cross-validation (GCV), the connection between splines and Bayesian posterior estimates, and "Wahba's problem," she has developed methods with applications in demographic studies, machine learning, DNA microarrays, risk modeling, medical imaging and climate prediction.

Grace grew up in the Washington, DC area and New Jersey, and graduated from Montclair High School. She was educated at Cornell (B.A. 1956), University of Maryland, College Park (M.A. 1962) and Stanford (Ph.D. 1966), and worked in industry for several years before receiving her doctorate in 1966 and settling in Madison in 1967. Although holding several visiting appointments, she has made Madison her home for over 50 years. She is the author of *Spline Models for Observational Data* which has garnered more than 8000 citations. Grace is treasured as an academic advisor and has mentored 39 Ph.D. students that have resulted in more than 330 academic descendants. She was elected to the United States National Academy of Sciences in 2000 and received an honorary degree of Doctor of Science from the University of Chicago in 2007. Wahba is a Fellow of several academic societies including the American Academy of Arts and Sciences, the American Association for the Advancement of Science, the American Statistical Association and the Institute of Mathematical Statistics. Over the years, she has received a selection of notable awards in the statistics community: R. A. Fisher Lectureship (2014), Gottfried E. Noether Senior Researcher Award (2009), Committee of Presidents of Statistical Societies Elizabeth Scott Award (1996) and the first Emanuel and Carol Parzen Prize for Statistical Innovation (1994).

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A Conversation with Francisco J. Samaniego

George G. Roussas and Debasis Bhattacharya

Abstract. In a wide-ranging interview, Professor George G. Roussas of the University of California, Davis, and Professor Debasis Bhatacharya of Visva-Bharati University, India, and a frequent visitor to UC Davis, engage Professor Francisco J. Samaniego in a discussion about his personal background, his education and the highlights of his professional career as a teacher and researcher.

Key words and phrases: Education, teaching and research, reliability theory, comparative statistical inference.

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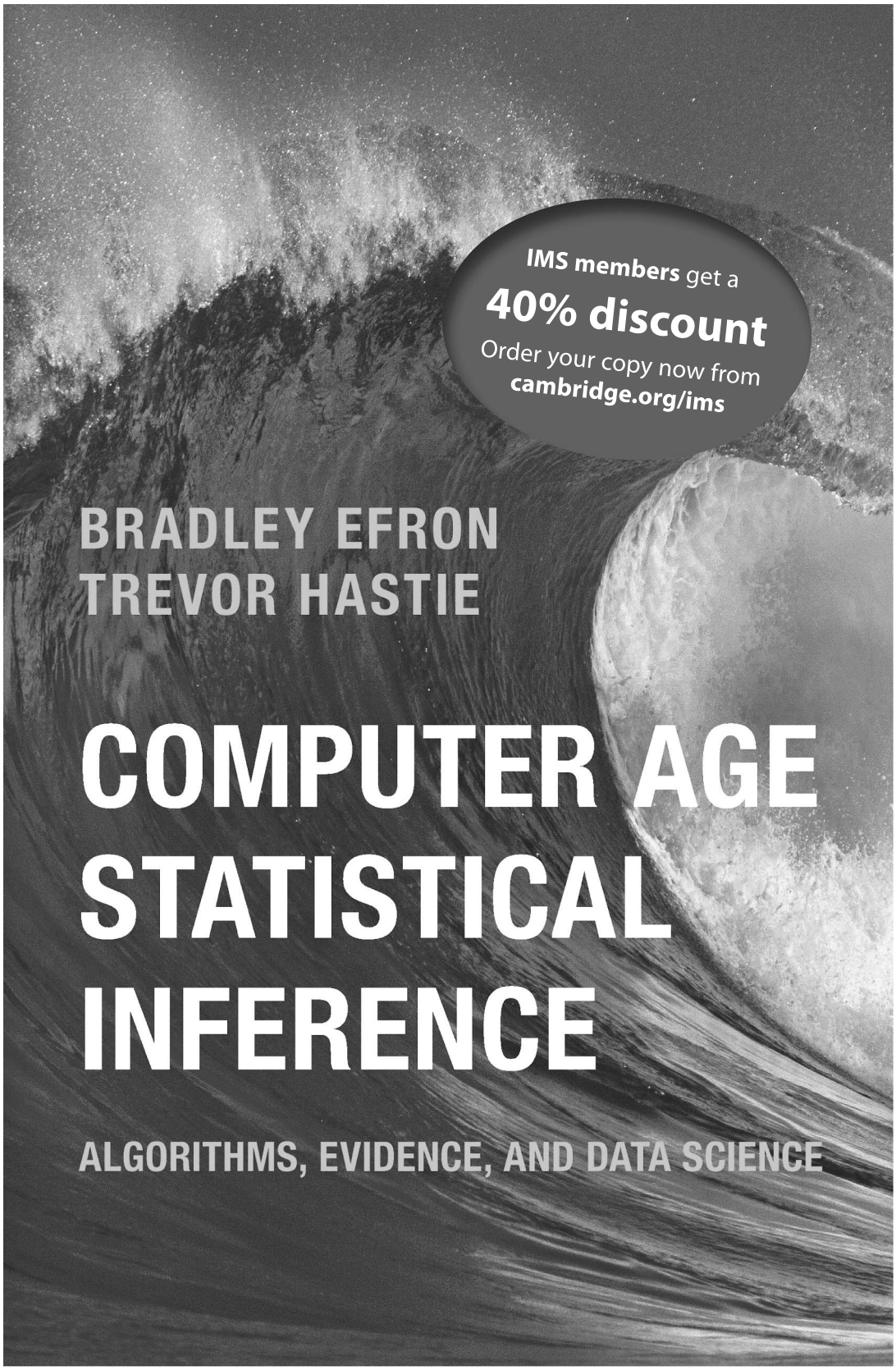
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