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Judicious Judgment Meets Unsettling Updating: Dilation, Sure Loss and Simpson's Paradox

Ruobin Gong and Xiao-Li Meng

Abstract. Imprecise probabilities alleviate the need for high-resolution and unwarranted assumptions in statistical modeling. They present an alternative strategy to reduce irreplicable findings. However, updating imprecise models requires the user to choose among alternative updating rules. Competing rules can result in incompatible inferences, and exhibit *dilation*, *contraction* and *sure loss*, unsettling phenomena that cannot occur with precise probabilities and the regular Bayes rule. We revisit some famous statistical paradoxes and show that the logical fallacy stems from a set of marginally plausible yet jointly incommensurable model assumptions akin to the trio of phenomena above. Discrepancies between the generalized Bayes ($\mathfrak{B}$) rule, Dempster's ($\mathfrak{D}$) rule and the Geometric ($\mathfrak{G}$) rule as competing updating rules for Choquet capacities of order 2 are discussed. We note that (1) $\mathfrak{B}$-rule cannot contract nor induce sure loss, but is the most prone to dilation due to "overfitting" in a certain sense; (2) in absence of prior information, both $\mathfrak{B}$- and $\mathfrak{G}$-rules are incapable to learn from data however informative they may be; (3) $\mathfrak{D}$- and $\mathfrak{G}$-rules can mathematically contradict each other by contracting while the other dilating. These findings highlight the invaluable role of judicious judgment in handling low-resolution information, and the care that needs to be take when applying updating rules to imprecise probability models.

Key words and phrases: Imprecise probability, model uncertainty, Choquet capacity, belief function, coherence, Monty Hall problem.

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Comment: On the History and Limitations of Probability Updating

Glenn Shafer

Abstract. Gong and Meng show that we can gain insights into classical paradoxes about conditional probability by acknowledging that apparently precise probabilities live within a larger world of imprecise probability. They also show that the notion of updating becomes problematic in this larger world. A closer look at the historical development of the notion of updating can give us further insights into its limitations.

Key words and phrases: Bayes's rule of conditioning, Dempster's rule, conditional probability, conditionalization, imprecise probabilities, probability protocols, relative probability, updating.

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Comment: Settle the Unsettling: An Inferential Models Perspective

Chuanhai Liu and Ryan Martin

Abstract. Here, we demonstrate that the *inferential model* (IM) framework, unlike the updating rules that Gong and Meng show to be unreliable, provides valid and efficient inferences/prediction while not being susceptible to sure loss. In this sense, the IM framework settles what Gong and Meng characterized as "unsettling."

Key words and phrases: Belief function, efficiency, lower and upper probability, inferential models, validity.

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Comment: Moving Beyond Sets of Probabilities

Gregory Wheeler

Abstract. The theory of lower previsions is designed around the principles of coherence and sure-loss avoidance, thus steers clear of all the updating anomalies highlighted in Gong and Meng's "Judicious Judgment Meets Unsettling Updating: Dilation, Sure Loss and Simpson's Paradox" except dilation. In fact, the traditional problem with the theory of imprecise probability is that coherent inference is too complicated rather than unsettling. Progress has been made simplifying coherent inference by demoting sets of probabilities from fundamental building blocks to secondary representations that are derived or discarded as needed.

Key words and phrases: Desirable gambles, lower previsions, imprecise probabilities, dilation.

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Comment: On Focusing, Soft and Strong Revision of Choquet Capacities and Their Role in Statistics

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Abstract. We congratulate Ruobin Gong and Xiao-Li Meng on their thought-provoking paper demonstrating the power of imprecise probabilities in statistics. In particular, Gong and Meng clarify important statistical paradoxes by discussing them in the framework of generalized uncertainty quantification and different conditioning rules used for updating. In this note, we characterize all three conditioning rules as envelopes of certain sets of conditional probabilities. This view also suggests some generalizations that can be seen as compromise rules. Similar to Gong and Meng, our derivations mainly focus on Choquet capacities of order 2, and so we also briefly discuss in general their role as statistical models. We conclude with some general remarks on the potential of imprecise probabilities to cope with the multidimensional nature of uncertainty.

Key words and phrases: Imprecise probabilities, Choquet capacities, updating, neighborhood models, generalized Bayes rule, Dempster's rule of conditioning.

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Rejoinder: Let's Be Imprecise in Order to Be Precise (About What We Don't Know)

Ruobin Gong and Xiao-Li Meng

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A Unified Primal Dual Active Set Algorithm for Nonconvex Sparse Recovery

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Abstract. In this paper, we consider the problem of recovering a sparse signal based on penalized least squares formulations. We develop a novel algorithm of primal-dual active set type for a class of nonconvex sparsity-promoting penalties, including ℓ^0, bridge, smoothly clipped absolute deviation, capped ℓ^1 and minimax concavity penalty. First, we establish the existence of a global minimizer for the related optimization problems. Then we derive a novel necessary optimality condition for the global minimizer using the associated thresholding operator. The solutions to the optimality system are coordinatewise minimizers, and under minor conditions, they are also local minimizers. Upon introducing the dual variable, the active set can be determined using the primal and dual variables together. Further, this relation lends itself to an iterative algorithm of active set type which at each step involves first updating the primal variable only on the active set and then updating the dual variable explicitly. When combined with a continuation strategy on the regularization parameter, the primal dual active set method is shown to converge globally to the underlying regression target under certain regularity conditions. Extensive numerical experiments with both simulated and real data demonstrate its superior performance in terms of computational efficiency and recovery accuracy compared with the existing sparse recovery methods.

Key words and phrases: Nonconvex penalty, sparsity, primal-dual active set algorithm, continuation, consistency.

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The Box–Cox Transformation: Review and Extensions

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Abstract. The Box–Cox power transformation family for nonnegative responses in linear models has a long and interesting history in both statistical practice and theory, which we summarize. The relationship between generalized linear models and log transformed data is illustrated. Extensions investigated include the transform both sides model and the Yeo–Johnson transformation for observations that can be positive or negative. The paper also describes an extended Yeo–Johnson transformation that allows positive and negative responses to have different power transformations. Analyses of data show this to be necessary. Robustness enters in the fan plot for which the forward search provides an ordering of the data. Plausible transformations are checked with an extended fan plot. These procedures are used to compare parametric power transformations with nonparametric transformations produced by smoothing.

Key words and phrases: ACE, AVAS, constructed variable, extended Yeo–Johnson transformation, forward search, linked plots, robust methods.

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Noncommutative Probability and Multiplicative Cascades

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Abstract. Various aspects of standard model particle physics might be explained by a suitably rich algebra acting on itself, as suggested by Furey (2015). The present paper develops the asymptotics of large causal tree diagrams that combine freely independent elements in such an algebra. The Marčenko–Pastur law and Wigner's semicircle law are shown to emerge as limits of normalized sum-over-paths of nonnegative elements assigned to the edges of causal trees. These results are established in the setting of noncommutative probability. Trees with classically independent positive edge weights (random multiplicative cascades) were originally proposed by Mandelbrot as a model displaying the fractal features of turbulence. The novelty of the present work is the use of noncommutative (free) probability to allow the edge weights to take values in an algebra. An application to theoretical neuroscience is also discussed.

Key words and phrases: Mandelbrot cascade, Marčenko–Pastur law, martingale convergence, random matrices, Wigner's semicircle law.

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A Selective Overview of Deep Learning

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Abstract. Deep learning has achieved tremendous success in recent years. In simple words, deep learning uses the composition of many nonlinear functions to model the complex dependency between input features and labels. While neural networks have a long history, recent advances have significantly improved their empirical performance in computer vision, natural language processing and other predictive tasks. From the statistical and scientific perspective, it is natural to ask: What is deep learning? What are the new characteristics of deep learning, compared with classical statistical methods? What are the theoretical foundations of deep learning?

To answer these questions, we introduce common neural network models (e.g., convolutional neural nets, recurrent neural nets, generative adversarial nets) and training techniques (e.g., stochastic gradient descent, dropout, batch normalization) from a statistical point of view. Along the way, we highlight new characteristics of deep learning (including depth and overparametrization) and explain their practical and theoretical benefits. We also sample recent results on theories of deep learning, many of which are only suggestive. While a complete understanding of deep learning remains elusive, we hope that our perspectives and discussions serve as a stimulus for new statistical research.

Key words and phrases: Neural networks, overparametrization, stochastic gradient descent, approximation theory, generalization error.

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Stochastic Approximation: From Statistical Origin to Big-Data, Multidisciplinary Applications

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Abstract. Stochastic approximation was introduced in 1951 to provide a new theoretical framework for root finding and optimization of a regression function in the then-nascent field of statistics. This review shows how it has evolved in response to other developments in statistics, notably time series and sequential analysis, and to applications in artificial intelligence, economics and engineering. Its resurgence in the big data era has led to new advances in both theory and applications of this microcosm of statistics and data science.

Key words and phrases: Control, gradient boosting, optimization, recursive stochastic algorithms, regret, weak greedy variable selection.

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Robust High-Dimensional Factor Models with Applications to Statistical Machine Learning

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Abstract. Factor models are a class of powerful statistical models that have been widely used to deal with dependent measurements that arise frequently from various applications from genomics and neuroscience to economics and finance. As data are collected at an ever-growing scale, statistical machine learning faces some new challenges: high dimensionality, strong dependence among observed variables, heavy-tailed variables and heterogeneity. High-dimensional robust factor analysis serves as a powerful toolkit to conquer these challenges.

This paper gives a selective overview on recent advance on high-dimensional factor models and their applications to statistics including Factor-Adjusted Robust Model selection (FarmSelect) and Factor-Adjusted Robust Multiple testing (FarmTest). We show that classical methods, especially principal component analysis (PCA), can be tailored to many new problems and provide powerful tools for statistical estimation and inference. We highlight PCA and its connections to matrix perturbation theory, robust statistics, random projection, false discovery rate, etc., and illustrate through several applications how insights from these fields yield solutions to modern challenges. We also present far-reaching connections between factor models and popular statistical learning problems, including network analysis and low-rank matrix recovery.

Key words and phrases: Factor model, PCA, covariance estimation, perturbation bounds, robustness, random sketch, FarmSelect, FarmTest.

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A Conversation with Dennis Cook

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Abstract. Dennis Cook is a Full Professor, School of Statistics, at the University of Minnesota. He received his BS degree in Mathematics from Northern Montana College, and MS and PhD degrees in Statistics from Kansas State University. He has served as Chair of the Department of Applied Statistics, Director of the Statistical Center and Director of the School of Statistics, all at the University of Minnesota.

His research areas include dimension reduction, linear and nonlinear regression, experimental design, statistical diagnostics, statistical graphics and population genetics. He has authored over 200 research articles and is author or co-author of two textbooks— An Introduction to Regression Graphics and Applied Regression Including Computing and Graphics—and three research monographs, Influence and Residuals in Regression, Regression Graphics: Ideas for Studying Regressions through Graphics and An Introduction to Envelopes: Dimension Reduction for Efficient Estimation in Multivariate Statistics.

He has served as Associate Editor of the *Journal of the American Statistical Association, The Journal of Quality Technology, Biometrika, Journal of the Royal Statistical Society* and *Statistica Sinica*. He is a four-time recipient of the Jack Youden Prize for Best Expository Paper in *Technometrics* as well as the Frank Wilcoxon Award for Best Technical Paper. He received the 2005 COPSS Fisher Lecture and Award, and he is a Fellow of the American Statistical Association and the Institute of Mathematical Statistics. The following conversation took place on March 22, 2019, following the banquet at the conference, "Cook's Distance and Beyond: A Conference Celebrating the Contributions of R. Dennis Cook." The interviewers were, Efstathia Bura (Effie), Bing Li, Lexin Li, Christopher Nachtsheim (Chris), Daniel Pena, Claude Messan Setodji and Robert Weiss (Rob).

Key words and phrases: Bayesian methods, central subspace, Cook's distance, envelope methods, influence, neural networks, optimal experimental design, regression diagnostics, sufficient dimension reduction.

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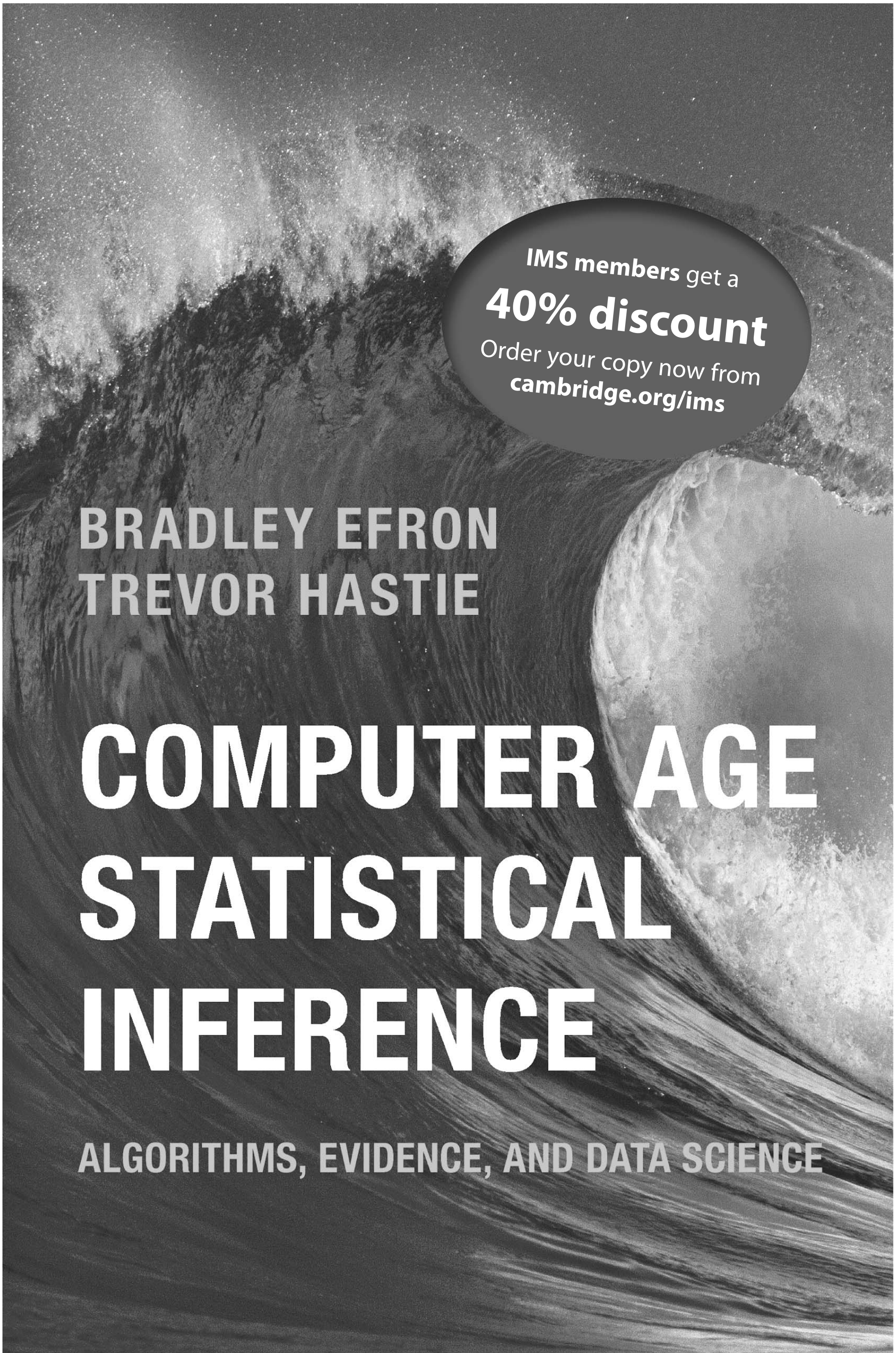
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