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Khinchin's 1929 Paper on Von Mises' Frequency Theory of Probability

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Abstract. In 1929, a few years prior to his colleague Kolmogorov's *Grundbegriffe*, the leading Russian probabilist Khinchin published a paper in which he commented on the foundational ambitions of von Mises' frequency theory of probability. This brief introduction provides background and context for the English translation of Khinchin's historically revealing paper, published as an online supplement.

Key words and phrases: Khinchin, Kolmogorov, von Mises, foundations of probability theory, frequency theory of probability, history of mathematics.

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A Problem in Forensic Science Highlighting the Differences between the Bayes Factor and Likelihood Ratio

Danica M. Ommen and Christopher P. Saunders

Abstract. This article is aimed at the growing number of statisticians interested in the important problem of interpreting evidence within the forensic identification of source problems. Our purpose is to formalize these forensic problems as statistical model selection problems. We use two different classes of statistics for quantifying the evidential value, the likelihood ratio and Bayes Factor. In forensics, both are commonly called the "likelihood ratio approach" and "the value of evidence" despite using different definitions of probability. In statistics, they are closely related to the traditional likelihood ratio from pattern recognition and the Bayes Factor used in model selection. For two different problem frameworks typical in forensic science, the common source and the specific source problems, we show the Bayes Factor and likelihood ratio are not equivalent, and highlight several interesting links between them. These contributions will help to elucidate the effects of choosing different definitions of probability when addressing the forensic identification of source problems. The broader population of statisticians may find this paper interesting as an introduction to forensic applications and for illuminating the connections between model selection methods from two different paradigms of statistics, particularly in view of the active recent discussions on the connections among Bayesian, Fiducial, Frequentist (BFF) approaches.

Key words and phrases: Bayesian, Frequentist, model selection, consistency, credible interval, forensics, common source, specific source.

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A Horse Race between the Block Maxima Method and the Peak–over–Threshold Approach

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Abstract. Classical extreme value statistics consists of two fundamental approaches: the block maxima (BM) method and the peak-over-threshold (POT) approach. It seems to be general consensus among researchers in the field that the POT approach makes use of extreme observations more efficiently than the BM method. We shed light on this discussion from three different perspectives. First, based on recent theoretical results for the BM method, we provide a theoretical comparison in i.i.d. scenarios. We argue that the data generating process may favour either one or the other approach. Second, if the underlying data possesses serial dependence, we argue that the choice of a method should be primarily guided by the ultimate statistical interest: for instance, POT is preferable for quantile estimation, while BM is preferable for return level estimation. Finally, we discuss the two approaches for multivariate observations and identify various open ends for future research.

Key words and phrases: Extreme value statistics, extreme value index, extremal index, stationary time series.

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A Hybrid Scan Gibbs Sampler for Bayesian Models with Latent Variables

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Abstract. Gibbs sampling is a widely popular Markov chain Monte Carlo algorithm that can be used to analyze intractable posterior distributions associated with Bayesian hierarchical models. There are two standard versions of the Gibbs sampler: The systematic scan (SS) version, where all variables are updated at each iteration, and the random scan (RS) version, where a single, randomly selected variable is updated at each iteration. The literature comparing the theoretical properties of SS and RS Gibbs samplers is reviewed, and an alternative *hybrid scan* Gibbs sampler is introduced, which is particularly well suited to Bayesian models with latent variables. The word "hybrid" reflects the fact that the scan used within this algorithm has both systematic and random elements. Indeed, at each iteration, one updates the entire set of latent variables, along with a randomly chosen block of the remaining variables. The hybrid scan (HS) Gibbs sampler has important advantages over the two standard scan Gibbs samplers. First, the HS algorithm is often easier to analyze from a theoretical standpoint. In particular, it can be much easier to establish the geometric ergodicity of a HS Gibbs Markov chain than to do the same for the corresponding SS and RS versions. Second, the sandwich methodology developed in (*Ann. Statist.* **36** (2008) 532–554), which is also reviewed, can be applied to the HS Gibbs algorithm (but not to the standard scan Gibbs samplers). It is shown that, under weak regularity conditions, adding sandwich steps to the HS Gibbs sampler always results in a theoretically superior algorithm. Three specific Bayesian hierarchical models of varying complexity are used to illustrate the results. One is a simple location-scale model for data from the Student's t distribution, which is used as a pedagogical tool. The other two are sophisticated, yet practical Bayesian regression models.

Key words and phrases: Geometric ergodicity, heavy-tailed errors, linear mixed model, Markov chain Monte Carlo, sandwich algorithm, shrinkage prior.

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Maximum Likelihood Multiple Imputation: Faster Imputations and Consistent Standard Errors Without Posterior Draws

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Abstract. Multiple imputation (MI) is a method for repairing and analyzing data with missing values. MI replaces missing values with a sample of random values drawn from an *imputation model*. The most popular form of MI, which we call *posterior draw multiple imputation* (PDMI), draws the parameters of the imputation model from a Bayesian posterior distribution. An alternative, which we call *maximum likelihood multiple imputation* (MLMI), estimates the parameters of the imputation model using maximum likelihood (or equivalent). Compared to PDMI, MLMI is faster and yields slightly more efficient point estimates.

A past barrier to using MLMI was the difficulty of estimating the standard errors of MLMI point estimates. We derive, implement and evaluate three consistent standard error formulas: (1) one combines variances within and between the imputed datasets, (2) one uses the score function and (3) one uses the bootstrap with two imputations of each bootstrapped sample. Formula (1) modifies for MLMI a formula that has long been used under PDMI, while formulas (2) and (3) can be used without modification under either PDMI or MLMI. We have implemented MLMI and the standard error estimators in the *mlmi* and *bootImpute* packages for R.

Key words and phrases: Missing data, incomplete data.

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Random Matrix Theory and Its Applications

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Abstract. This article reviews the important ideas behind random matrix theory (RMT), which has become a major tool in a variety of disciplines, including mathematical physics, number theory, combinatorics and multivariate statistical analysis. Much of the theory involves ensembles of random matrices that are governed by some probability distribution. Examples include Gaussian ensembles and Wishart–Laguerre ensembles. Interest has centered on studying the spectrum of random matrices, especially the extreme eigenvalues, suitably normalized, for a single Wishart matrix and for two Wishart matrices, for finite and infinite sample sizes in the real and complex cases. The Tracy–Widom Laws for the probability distribution of a normalized largest eigenvalue of a random matrix have become very prominent in RMT. Limiting probability distributions of eigenvalues of a certain random matrix lead to Wigner's Semicircle Law and Marčenko–Pastur's Quarter-Circle Law. Several applications of these results in RMT are described in this article.

Key words and phrases: Eigenvalue density, Gaussian ensembles, Jacobi ensembles, Marčenko–Pastur's quarter-circle law, Spiked covariance model, Tracy–Widom laws, Wigner matrix, Wigner's semicircle law, Wishart matrix, Wishart–Laguerre ensembles.

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The GENIUS Approach to Robust Mendelian Randomization Inference

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Abstract. Mendelian randomization (MR) is a popular instrumental variable (IV) approach, in which one or several genetic markers serve as IVs that can sometimes be leveraged to recover valid inferences about a given exposure-outcome causal association subject to unmeasured confounding. A key IV identification condition known as the exclusion restriction states that the IV cannot have a direct effect on the outcome which is not mediated by the exposure in view. In MR studies, such an assumption requires an unrealistic level of prior knowledge about the mechanism by which genetic markers causally affect the outcome. As a result, possible violation of the exclusion restriction can seldom be ruled out in practice. To address this concern, we introduce a new class of IV estimators which are robust to violation of the exclusion restriction under data generating mechanisms commonly assumed in MR literature. The proposed approach named “MR G-Estimation under No Interaction with Unmeasured Selection” (MR GENIUS) improves on Robins’ G-estimation by making it robust to both additive unmeasured confounding and violation of the exclusion restriction assumption. In certain key settings, MR GENIUS reduces to the estimator of Lewbel (*J. Bus. Econom. Statist.* **30** (2012) 67–80) which is widely used in econometrics but appears largely unappreciated in MR literature. More generally, MR GENIUS generalizes Lewbel’s estimator to several key practical MR settings, including multiplicative causal models for binary outcome, multiplicative and odds ratio exposure models, case control study design and censored survival outcomes.

Key words and phrases: Additive model, confounding, exclusion restriction, G-estimation, instrumental variable, robustness.

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A General Framework for the Analysis of Adaptive Experiments

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Abstract. Adaptive experiments have design features that adapt to the accumulating data and are therefore informative about the parameter of interest. As a consequence, the overall information in an adaptive experiment is a combination of information from two sources, the realized design and the observed outcomes. This paper presents a general framework for the analysis of adaptive experiments, based on the decomposition of overall information into design information and outcome information. Likelihood inference is discussed, beginning with assumptions that guarantee insensitivity of the likelihood to the adaptive design. We then focus on the relative merits of unconditional and conditional inference. Although conditional inference is inefficient due to the nonancillary design, unconditional inference may be biased conditional on the realized design. Identifying such conditional bias in a given experiment is a motivation of the proposed framework. We show that conditional bias stems from correlation between the total information and the design information, and that this bias is most pronounced in samples where the design information is inconsistent with the outcome information. Thus, by viewing the unconditional likelihood as the aggregation of information from a design likelihood and a conditional likelihood, we can use meta-analysis principles to assess heterogeneity between the two information sources. When such heterogeneity is detected, conditional inference may be more appropriate. Interpretation from a Bayesian perspective is also discussed.

Key words and phrases: Adaptive design, Bayesian inference, conditional inference, estimation bias, randomized experiment, sequential experiment.

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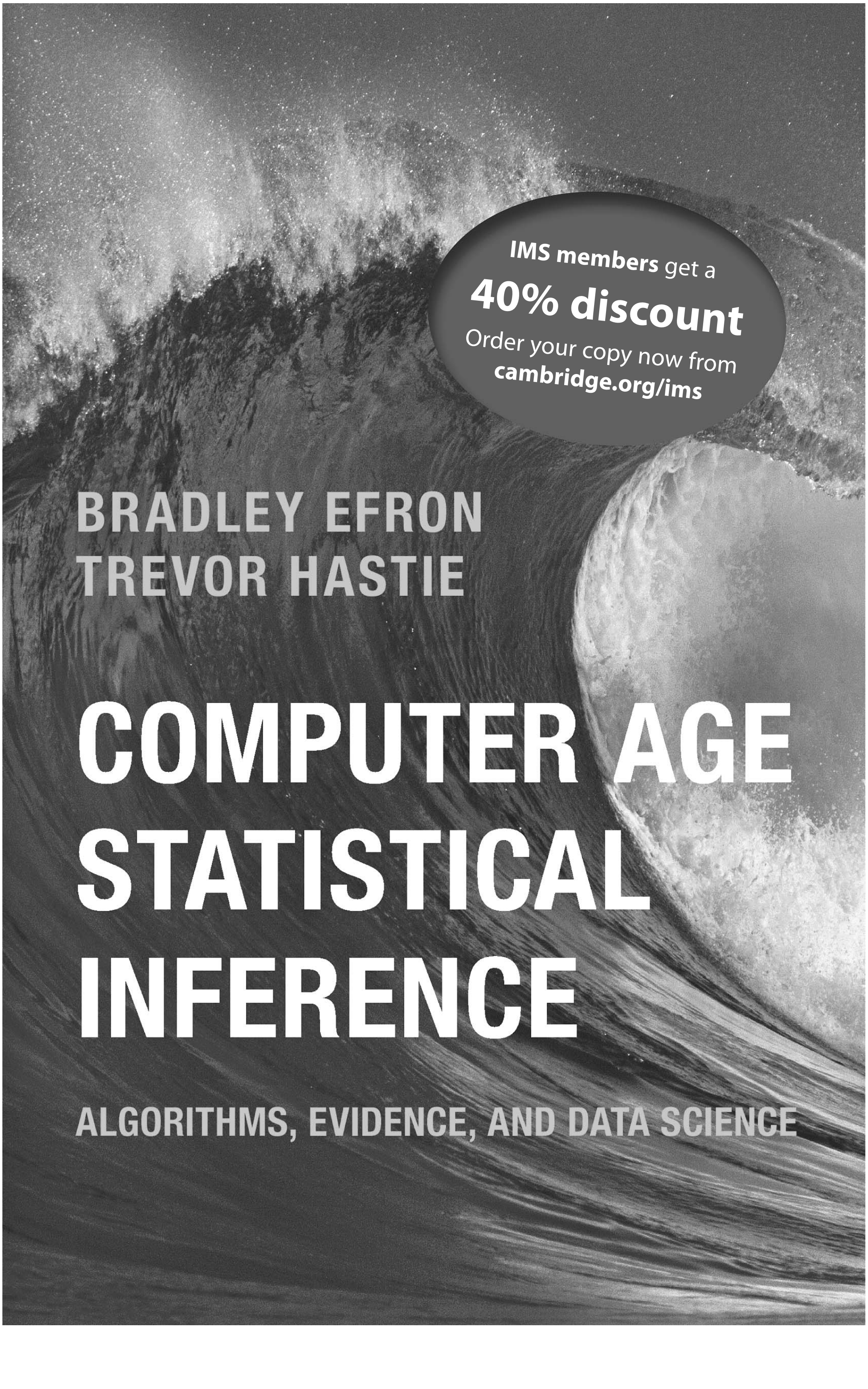
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