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Interpreting p-Values and Confidence Intervals Using Well-Calibrated Null Preference Priors

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Abstract. We propose well-calibrated null preference priors for use with one-sided hypothesis tests, such that resulting Bayesian and frequentist inferences agree. Null preference priors mean that they have nearly 100% of their prior belief in the null hypothesis, and well-calibrated priors mean that the resulting posterior beliefs in the alternative hypothesis are not overconfident. This formulation expands the class of problems giving Bayes-frequentist agreement to include problems involving discrete distributions such as binomial and negative binomial one- and two-sample exact (i.e., valid) tests. When applicable, these priors give posterior belief in the null hypothesis that is a valid p-value, and the null preference prior emphasizes that large p-values may simply represent insufficient data to overturn prior belief. This formulation gives a Bayesian interpretation of some common frequentist tests, as well as more intuitively explaining lesser known and less straightforward confidence intervals for two-sample tests.

Key words and phrases: Bayesian calibration, Bayesian hypothesis testing, frequentist matching priors, objective priors, probability matching priors.

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Measurement Error Models: From Nonparametric Methods to Deep Neural Networks

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Abstract. The success of deep learning has inspired a lot of recent interests in exploiting neural network structures for statistical inference and learning. In this paper, we review some popular deep neural network structures and techniques under the framework of nonparametric regression with measurement errors. In particular, we demonstrate how to use a fully connected feed-forward neural network to approximate the regression function $f(x)$, explain how to use a normalizing flow to approximate the prior distribution of X, and detail how to construct an inference network to generate approximate posterior samples of X. After reviewing recent advances in variational inference for deep neural networks, such as the importance weighted autoencoder, doubly reparametrized gradient estimator, and nonlinear independent components estimation, we describe an inference procedure built upon these advances. An extensive numerical study is presented to compare the neural network approach with classical nonparametric methods, which suggests that the neural network approach is more flexible in accommodating different classes of regression functions and performs superior or comparable to the best available method in many settings.

Key words and phrases: Doubly reparametrized gradient estimator, error-in-variables, fully connected feed-forward neural network, importance weighted autoencoder, normalizing flow, SIMEX, variational autoencoder.

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High-Performance Statistical Computing in the Computing Environments of the 2020s

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Abstract. Technological advances in the past decade, hardware and software alike, have made access to high-performance computing (HPC) easier than ever. We review these advances from a statistical computing perspective. Cloud computing makes access to supercomputers affordable. Deep learning software libraries make programming statistical algorithms easy and enable users to write code once and run it anywhere—from a laptop to a workstation with multiple graphics processing units (GPUs) or a supercomputer in a cloud. Highlighting how these developments benefit statisticians, we review recent optimization algorithms that are useful for high-dimensional models and can harness the power of HPC. Code snippets are provided to demonstrate the ease of programming. We also provide an easy-to-use distributed matrix data structure suitable for HPC. Employing this data structure, we illustrate various statistical applications including large-scale positron emission tomography and ℓ_1-regularized Cox regression. Our examples easily scale up to an 8-GPU workstation and a 720-CPU-core cluster in a cloud. As a case in point, we analyze the onset of type-2 diabetes from the UK Biobank with 200,000 subjects and about 500,000 single nucleotide polymorphisms using the HPC ℓ_1-regularized Cox regression. Fitting this half-million-variate model takes less than 45 minutes and reconfirms known associations. To our knowledge, this is the first demonstration of the feasibility of penalized regression of survival outcomes at this scale.

Key words and phrases: High-performance statistical computing, graphics processing units (GPUs), cloud computing, deep learning, MM algorithms, ADMM, PDHG, Cox regression.

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The SPDE Approach to Matérn Fields: Graph Representations

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Abstract. This paper investigates Gaussian Markov random field approximations to nonstationary Gaussian fields using graph representations of stochastic partial differential equations. We establish approximation error guarantees building on the theory of spectral convergence of graph Laplacians. The proposed graph representations provide a generalization of the Matérn model to unstructured point clouds, and facilitate inference and sampling using linear algebra methods for sparse matrices. In addition, they bridge and unify several models in Bayesian inverse problems, spatial statistics and graph-based machine learning. We demonstrate through examples in these three disciplines that the unity revealed by graph representations facilitates the exchange of ideas across them.

Key words and phrases: Gauss Matérn fields, Gaussian Markov random fields, graph Laplacians, latent Gaussian models, stochastic partial differential equations.

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The Covariate-Adjusted ROC Curve: The Concept and Its Importance, Review of Inferential Methods, and a New Bayesian Estimator

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Abstract. Accurate diagnosis of disease is of fundamental importance in clinical practice and medical research. Before a medical diagnostic test is routinely used in practice, its ability to distinguish between diseased and nondiseased states must be rigorously assessed. The receiver operating characteristic (ROC) curve is the most popular used tool for evaluating the diagnostic accuracy of continuous-outcome tests. It has been acknowledged that several factors (e.g., subject-specific characteristics such as age and/or gender) can affect the test outcomes and accuracy beyond disease status. Recently, the covariate-adjusted ROC curve has been proposed and successfully applied as a global summary measure of diagnostic accuracy that takes covariate information into account. The aim of this paper is three-fold. First, we motivate the importance of including covariate-information, whenever available, in ROC analysis and, in particular, how the covariate-adjusted ROC curve is an important tool in this context. Second, we review and provide insight on the existing approaches for estimating the covariate-adjusted ROC curve. Third, we develop a highly flexible Bayesian method, based on the combination of a Dirichlet process mixture of additive normal models and the Bayesian bootstrap, for conducting inference about the covariate-adjusted ROC curve. A simulation study is conducted to assess the performance of the different methods and it also demonstrates the ability of our proposed Bayesian model to successfully recover the true covariate-adjusted ROC curve and to produce valid inferences in a variety of complex scenarios. The methods are applied to an endocrine study where the goal is to assess the accuracy of the body mass index, adjusted for age and gender, for detecting clusters of cardiovascular disease risk factors.

Key words and phrases: Classification accuracy, covariate-adjustment, decision threshold, diagnostic test, Dirichlet process (mixture) model, receiver operating characteristic curve.

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A Regression Perspective on Generalized Distance Covariance and the Hilbert–Schmidt Independence Criterion

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Abstract. In a seminal paper, Sejdinovic et al. (*Ann. Statist.* **41** (2013) 2263–2291) showed the equivalence of the Hilbert–Schmidt Independence Criterion (HSIC) and a generalization of distance covariance. In this paper, the two notions of dependence are unified with a third prominent concept for independence testing, the "global test" introduced in (*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **68** (2006) 477–493). The new viewpoint provides novel insights into all three test traditions, as well as a unified overall view of the way all three tests contrast with classical association tests. As our main result, a regression perspective on HSIC and generalized distance covariance is obtained, allowing such tests to be used with nuisance covariates or for survival data. Several more examples of cross-fertilization of the three traditions are provided, involving theoretical results and novel methodology. To illustrate the difference between classical statistical tests and the unified HSIC/distance covariance/global tests we investigate the case of association between two categorical variables in depth.

Key words and phrases: Distance covariance, distance correlation, Hilbert–Schmidt Independence Criterion, global test, equivalence, locally most powerful.

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Methods to Compute Prediction Intervals: A Review and New Results

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Abstract. The purpose of this paper is to review both classic and modern methods for constructing prediction intervals. We focus, primarily, on model-based non-Bayesian methods for the prediction of a scalar random variable, but we also include Bayesian methods with objective prior distributions. Our review of non-Bayesian methods follows two lines: general methods based on (approximate) pivotal quantities and methods based on non-Bayesian predictive distributions. The connection between these two types of methods is described for distributions in the (log-)location-scale family. We also discuss extending the general prediction methods to data with complicated dependence structures as well as some nonparametric prediction methods (e.g., conformal prediction).

Key words and phrases: Bootstrap, calibration, coverage probability, prediction interval, predictive distribution.

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Approximate Confidence Intervals for a Binomial p—Once Again

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Abstract. The problem of constructing a reasonably simple yet well-behaved confidence interval for a binomial parameter p is old but still fascinating and surprisingly complex. During the last century, many alternatives to the poorly behaved standard Wald interval have been suggested. It seems though that the Wald interval is still much in use in spite of many efforts over the years through publications to point out its deficiencies. This paper constitutes yet another attempt to provide an alternative and it builds on a special case of a general technique for adjusted intervals primarily based on Wald type statistics. The main idea is to construct an approximate pivot with uncorrelated, or nearly uncorrelated, components. The resulting AN (Andersson–Nerman) interval, as well as a modification thereof, is compared with the well-renowned Wilson and AC (Agresti–Coull) intervals and the subsequent discussion will in itself hopefully shed some new light on this seemingly elementary interval estimation situation. Generally, an alternative to the Wald interval is to be judged not only by performance, its expression should also indicate *why* we will obtain a better behaved interval. It is argued that the well-behaved AN interval meets this requirement.

Key words and phrases: Correlation, coverage probability, score statistic, skewness, Wald statistic.

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A Conversation with David J. Aldous

Shankar Bhamidi

Abstract. David John Aldous was born in Exeter U.K. on July 13, 1952. He received a B.A. and Ph.D. in Mathematics in 1973 and 1977, respectively from Cambridge. After spending two years as a research fellow at St. John's College, Cambridge, he joined the Department of Statistics at the University of California, Berkeley in 1979 where he spent the rest of his academic career until retiring in 2018. He is known for seminal contributions on many topics within probability including weak convergence and tightness, exchangeability, Markov chain mixing times, Poisson clumping heuristic and limit theory for large discrete random structures including random trees, stochastic coagulation and fragmentation systems, models of complex networks and interacting particle systems on such structures. For his contributions to the field, he has received numerous honors and awards including the Rollo Davidson prize in 1980, the inaugural Loeve prize in Probability in 1993, and the Brouwer medal in 2021, and being elected as an IMS fellow in 1985, Fellow of the Royal Society in 1994, Fellow of the American Academy of Arts and Sciences in 2004, elected to the National Academy of Sciences (foreign associate) in 2010, ICM plenary speaker in 2010 and AMS fellow in 2012.

Key words and phrases: Exchangeability, Markov chain mixing times, scaling limits, local weak convergence, random graphs, network models.

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Comments on *Confidence as Likelihood* by Pawitan and Lee in *Statistical Science*, November 2021

Michael Lavine and Jan F. Bjørnstad

Abstract. Pawitan and Lee (*Statist. Sci.* **36** (2021) 509–517) attempt to show a correspondence between confidence and likelihood, specifically, that "confidence is in fact an extended likelihood" (*Statist. Sci.* **36** (2021) 509–517, abstract). The word "extended" means that the likelihood function can accommodate unobserved random variables such as random effects and future values; see (*J. Amer. Statist. Assoc.* **91** (1996) 791–806) for details. Here, we argue that the extended likelihood presented by (*Statist. Sci.* **36** (2021) 509–517) is not the correct extended likelihood and does not justify interpreting confidence as likelihood.

Key words and phrases: Likelihood, confidence.

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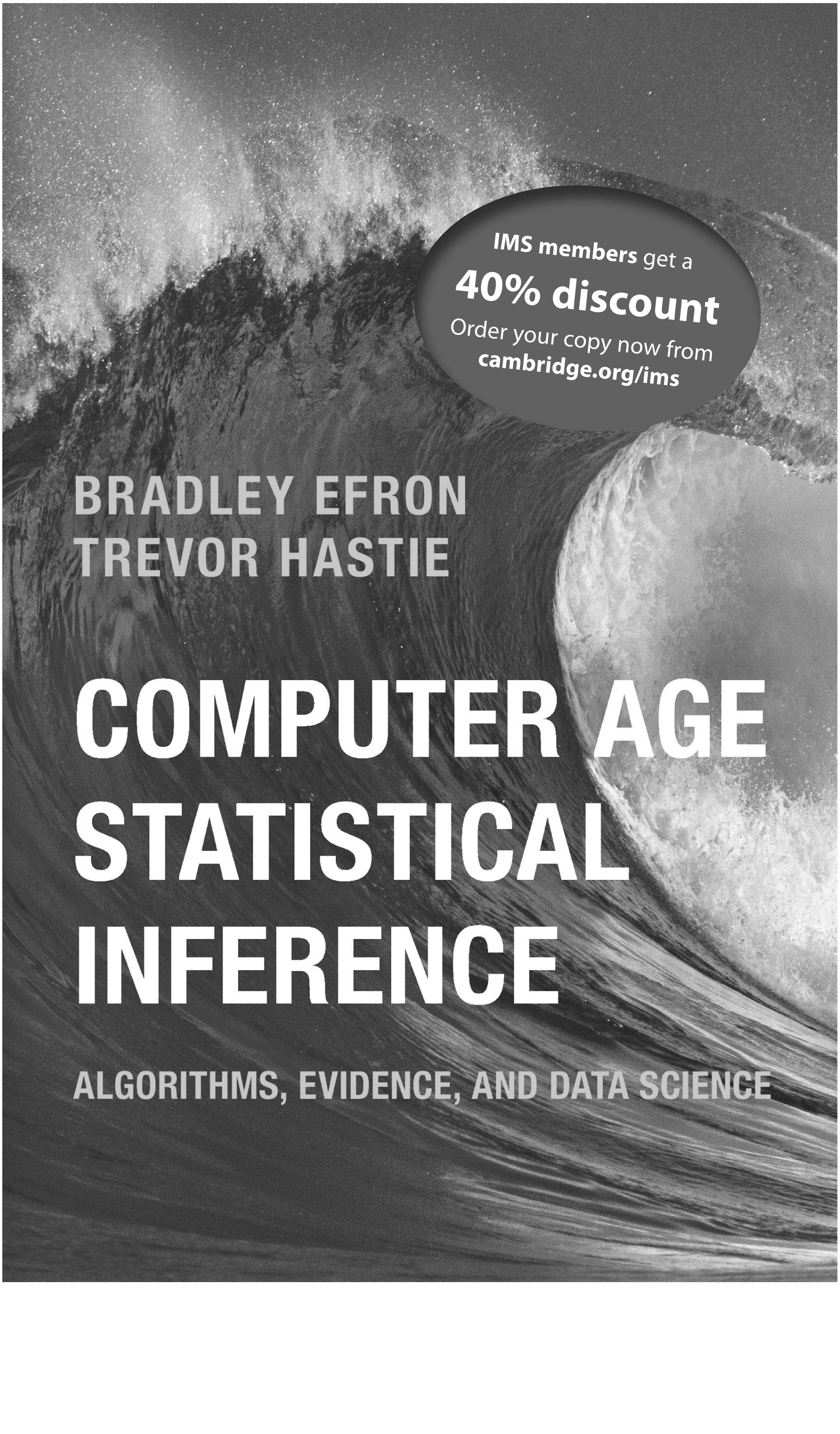
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