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The Costs and Benefits of Uniformly Valid Causal Inference with High-Dimensional Nuisance Parameters

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Abstract. Important advances have recently been achieved in developing procedures yielding uniformly valid inference for a low dimensional causal parameter when high-dimensional nuisance models must be estimated. In this paper, we review the literature on uniformly valid causal inference and discuss the costs and benefits of using uniformly valid inference procedures. Naive estimation strategies based on regularization, machine learning, or a preliminary model selection stage for the nuisance models have finite sample distributions which are badly approximated by their asymptotic distributions. To solve this serious problem, estimators which converge uniformly in distribution over a class of data generating mechanisms have been proposed in the literature. In order to obtain uniformly valid results in high-dimensional situations, sparsity conditions for the nuisance models need typically to be made, although a double robustness property holds, whereby if one of the nuisance model is more sparse, the other nuisance model is allowed to be less sparse. While uniformly valid inference is a highly desirable property, uniformly valid procedures pay a high price in terms of inflated variability. Our discussion of this dilemma is illustrated by the study of a double-selection outcome regression estimator, which we show is uniformly asymptotically unbiased, but is less variable than uniformly valid estimators in the numerical experiments conducted.

Key words and phrases: Double robustness, machine learning, post-model selection inference, regularization, superefficiency.

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Additive Bayesian Variable Selection under Censoring and Misspecification

David Rossell and Francisco Javier Rubio

Abstract. We discuss the role of misspecification and censoring on Bayesian model selection in the contexts of right-censored survival and concave log-likelihood regression. Misspecification includes wrongly assuming the censoring mechanism to be noninformative. Emphasis is placed on additive accelerated failure time, Cox proportional hazards and probit models. We offer a theoretical treatment that includes local and nonlocal priors, and a general nonlinear effect decomposition to improve power-sparsity trade-offs. We discuss a fundamental question: what solution can one hope to obtain when (inevitably) models are misspecified, and how to interpret it? Asymptotically, covariates that do not have predictive power for neither the outcome nor (for survival data) censoring times, in the sense of reducing a likelihood-associated loss, are discarded. Misspecification and censoring have an asymptotically negligible effect on false positives, but their impact on power is exponential. We show that it can be advantageous to consider simple models that are computationally practical yet attain good power to detect potentially complex effects, including the use of finite-dimensional basis to detect truly nonparametric effects. We also discuss algorithms to capitalize on sufficient statistics and fast likelihood approximations for Gaussian-based survival and binary models.

Key words and phrases: Additive regression, generalized additive model, misspecification, model selection, survival.

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On Some Connections Between Esscher's Tilting, Saddlepoint Approximations, and Optimal Transportation: A Statistical Perspective

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Abstract. We showcase some unexplored connections between saddlepoint approximations, measure transportation, and some key topics in information theory. To bridge these different areas, we review selectively the fundamental results available in the literature and we draw the connections between them. First, for a generic random variable we explain how the Esscher's tilting (which is a result rooted in information theory and lies at the heart of saddlepoint approximations) is connected to the solution of the dual Kantorovich problem (which lies at the heart of measure transportation theory) via the Legendre transform of the cumulant generating function. Then, we turn to statistics: we illustrate the connections when the random variable we work with is the sample mean or a statistic with known (either exact or approximate) cumulant generating function. The unveiled connections offer the possibility to look at the saddlepoint approximations from different angles, putting under the spotlight the links to convex analysis (via the notion of duality) or differential geometry (via the notion of geodesic). We feel these possibilities can trigger a knowledge transfer between statistics and other disciplines, like mathematics and machine learning. A discussion on some topics for future research concludes the paper.

Key words and phrases: Change of variable, Kullback–Leibler divergence, geodesic, optimal transportation map, Wasserstein distance.

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Bayesian Adaptive Randomization with Compound Utility Functions

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Abstract. Bayesian adaptive designs formalize the use of previous knowledge at the planning stage of an experiment, permitting recursive updating of the prior information. They often make use of utility functions, while also allowing for randomization. We review frequentist and Bayesian adaptive design methods and show that some of the frequentist adaptive design methodology can also be employed in a Bayesian context. We use compound utility functions for the Bayesian designs, that are a trade-off between an optimal design information criterion, that represents the acquisition of scientific knowledge, and some ethical or utilitarian gain. We focus on binary response models on two groups with independent Beta prior distributions on the success probabilities. The treatment allocation is shown to converge to the allocation that produces the maximum utility. Special cases are the Bayesian Randomized (simply) Adaptive Compound (BRAC) design, an extension of the frequentist Sequential Maximum Likelihood (SML) design and the Bayesian Randomized (doubly) Adaptive Compound Efficient (BRACE) design, a generalization of the Efficient Randomized Adaptive DEsign (ERADE). Numerical simulation studies compare BRAC with BRACE when D-optimality is the information criterion chosen. In analogy with the frequentist theory, the BRACE-D design appears more efficient than the BRAC-D design.

Key words and phrases: Bayesian designs, binary response model, doubly adaptive designs, optimal design criteria, optimal target, utility functions.

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Double-Estimation-Friendly Inference for High-Dimensional Misspecified Models

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Abstract. All models may be wrong—but that is not necessarily a problem for inference. Consider the standard t-test for the significance of a variable X for predicting response Y while controlling for p other covariates Z in a random design linear model. This yields correct asymptotic type I error control for the null hypothesis that X is conditionally independent of Y given Z under an *arbitrary* regression model of Y on (X, Z), provided that a linear regression model for X on Z holds. An analogous robustness to misspecification, which we term the "double-estimation-friendly" (DEF) property, also holds for Wald tests in generalised linear models, with some small modifications.

In this expository paper, we explore this phenomenon, and propose methodology for high-dimensional regression settings that respects the DEF property. We advocate specifying (sparse) generalised linear regression models for both Y and the covariate of interest X; our framework gives valid inference for the conditional independence null if either of these hold. In the special case where both specifications are linear, our proposal amounts to a small modification of the popular debiased Lasso test. We also investigate constructing confidence intervals for the regression coefficient of X via inverting our tests; these have coverage guarantees even in partially linear models where the contribution of Z to Y can be arbitrary. Numerical experiments demonstrate the effectiveness of the methodology.

Key words and phrases: Conditional independence, high-dimensional inference, Debiased Lasso, generalised linear models, double robustness.

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Randomization-Based Test for Censored Outcomes: A New Look at the Logrank Test

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Abstract. Two-sample tests with censored outcomes are a classical topic in statistics with wide use even in cutting edge applications. There are at least two modes of inference used to justify two-sample tests. One is usual superpopulation inference assuming that units are independent and identically distributed (i.i.d.) samples from some superpopulation; the other is finite population inference that relies on the random assignments of units into different groups. When randomization is actually implemented, the latter has the advantage of avoiding distributional assumptions on the outcomes. In this paper, we focus on finite population inference for censored outcomes, which has been less explored in the literature. Moreover, we allow the censoring time to depend on treatment assignment, under which exact permutation inference is unachievable. We find that, surprisingly, the usual logrank test can also be justified by randomization. Specifically, under a Bernoulli randomized experiment with noninformative i.i.d. censoring, the logrank test is asymptotically valid for testing Fisher's null hypothesis of no treatment effect on any unit. The asymptotic validity of the logrank test does not require any distributional assumption on the potential event times. We further extend the theory to the stratified logrank test, which is useful for randomized block designs and when censoring mechanisms vary across strata. In sum, the developed theory for the logrank test from finite population inference supplements its classical theory from usual superpopulation inference, and helps provide a broader justification for the logrank test.

Key words and phrases: Potential outcome, potential censoring time, design-based inference, noninformative censoring, stratified logrank test.

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The Poisson Binomial Distribution—Old & New

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Abstract. This is an expository article on the Poisson binomial distribution. We review lesser known results and recent progress on this topic, including geometry of polynomials and distribution learning. We also provide examples to illustrate the use of the Poisson binomial machinery. Some open questions of approximating rational fractions of the Poisson binomial are presented.

Key words and phrases: Distribution learning, geometry of polynomials, Poisson binomial distribution, Poisson/normal approximation, stochastic ordering, strong Rayleigh property.

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Stein's Method Meets Computational Statistics: A Review of Some Recent Developments

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Abstract. Stein's method compares probability distributions through the study of a class of linear operators called Stein operators. While mainly studied in probability and used to underpin theoretical statistics, Stein's method has led to significant advances in computational statistics in recent years. The goal of this survey is to bring together some of these recent developments, and in doing so, to stimulate further research into the successful field of Stein's method and statistics. The topics we discuss include tools to benchmark and compare sampling methods such as approximate Markov chain Monte Carlo, deterministic alternatives to sampling methods, control variate techniques, parameter estimation and goodness-of-fit testing.

Key words and phrases: Stein's method, sample quality, approximate Markov chain Monte Carlo, variational inference, control variates, goodness-of-fit testing, maximum likelihood estimator, likelihood ratio, prior sensitivity.

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Local scale invariance and robustness of proper scoring rules

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Abstract. Averages of proper scoring rules are often used to rank probabilistic forecasts. In many cases, the individual terms in these averages are based on observations and forecasts from different distributions. We show that some of the most popular proper scoring rules, such as the continuous ranked probability score (CRPS), give more importance to observations with large uncertainty, which can lead to unintuitive rankings. To describe this issue, we define the concept of local scale invariance for scoring rules. A new class of generalized proper kernel scoring rules is derived and as a member of this class we propose the scaled CRPS (SCRPS). This new proper scoring rule is locally scale invariant and, therefore, works in the case of varying uncertainty. Like the CRPS, it is computationally available for output from ensemble forecasts, and does not require the ability to evaluate densities of forecasts.

We further define robustness of scoring rules, show why this also can be an important concept for average scores unless one is specifically interested in extremes, and derive new proper scoring rules that are robust against outliers. The theoretical findings are illustrated in three different applications from spatial statistics, stochastic volatility models and regression for count data.

Key words and phrases: Probabilistic forecasting, model selection, spatial statistics, forecast ranking.

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In Praise (and Search) of J. V. Uspensky

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Abstract. The two of us have shared a fascination with James Victor Uspensky's 1937 textbook *Introduction to Mathematical Probability* ever since our graduate student days: it contains many interesting results not found in other books on the same subject in the English language, together with many non-trivial examples, all clearly stated with careful proofs. We present some of Uspensky's gems to a modern audience hoping to tempt others to read Uspensky for themselves, as well as report on a few of the other mathematical topics he also wrote about (e.g., his book on number theory contains early results about perfect shuffles).

Uspensky led an interesting life: a member of the Russian Academy of Sciences, he spoke at the 1924 International Congress of Mathematicians in Toronto before leaving Russia in 1929 and coming to the US and Stanford. Comparatively little has been written about him in English; the second half of this paper attempts to remedy this.

Key words and phrases: J. V. Uspensky, A. A. Markov, probability theory, Markov's method of continued fractions, Bernoulli's theorem, Lexis ratio, card shuffling, Russian mathematics, Stanford Mathematics Department.

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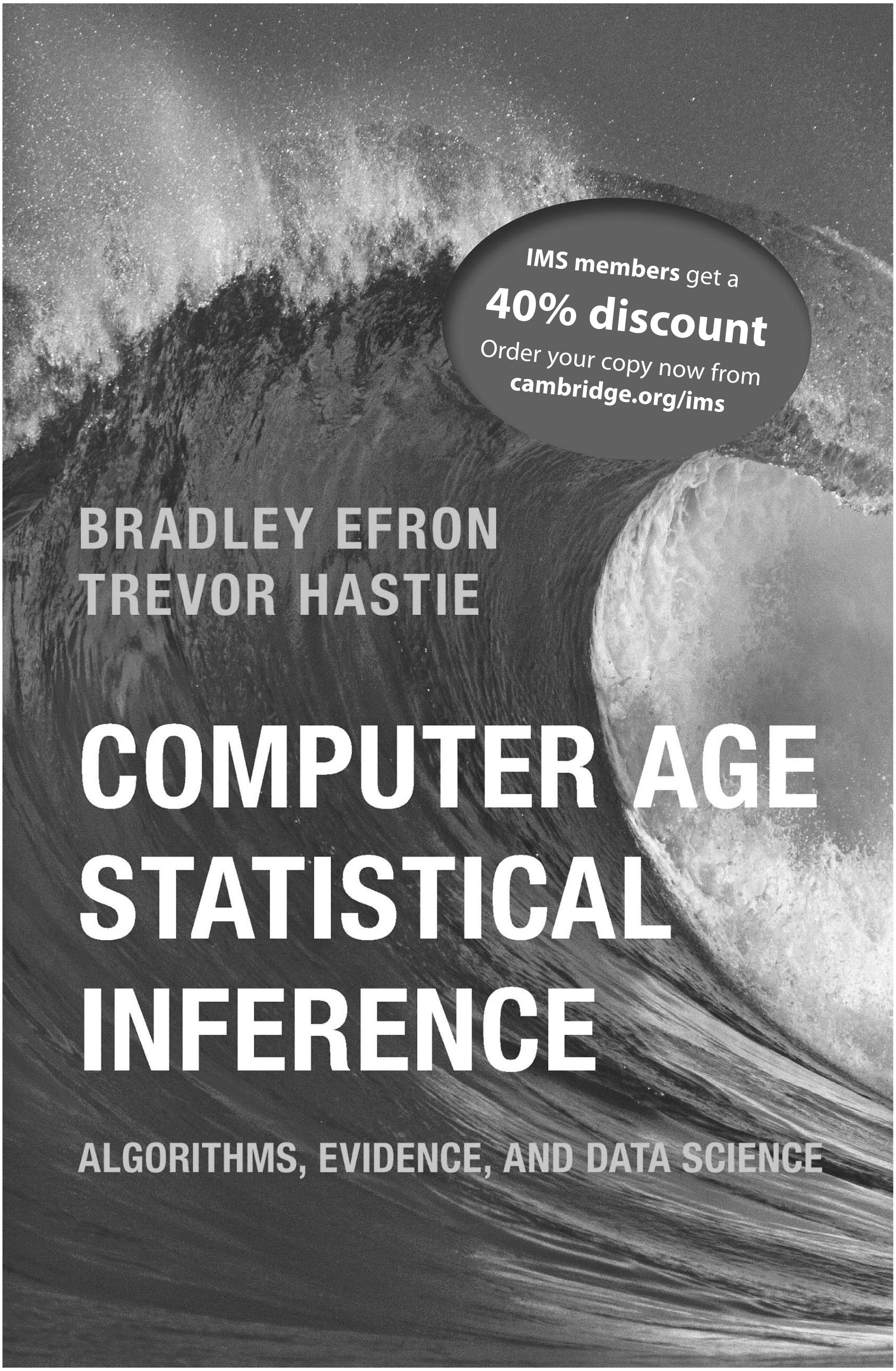

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