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Introduction to the Special Issue on Contemporary Bayesian Prediction

Bertrand Clarke and Pietro Rigo

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A Cheat Sheet for Bayesian Prediction

Bertrand Clarke and Yuling Yao

Abstract. This paper reviews the growing field of Bayesian prediction. Bayesian point and interval prediction are defined and situated in statistical prediction more generally. Then, four general approaches to Bayesian prediction are described and we turn to predictor selection. This can be done predictively or nonpredictively; predictors can be based on single models or multiple models. We call these unitary predictors and model average predictors, respectively. Predictors can also be explanatory meaning they are based on physical modeling or algorithmic meaning they are regarded simply as input-output relations. Then we turn to aspects of prediction in the context of large observational data sets, the most recent topic to emerge in prediction. We conclude with a summary and statement of several open problems.

Key words and phrases: Predictive density, calibration, validation, predictor selection, predictor evaluation, Bayes rule, Ionescu Tulcea theorem, decision theory, prequential, model average, median model, streaming data.

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A Probabilistic View on Predictive Constructions for Bayesian Learning

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Abstract. Given a sequence $X = (X_1, X_2, \dots)$ of random observations, a Bayesian forecaster aims to predict X_{n+1} based on $(X_1, \dots, X_n)$ for each $n \geq 0$. To this end, in principle, she only needs to select a collection $\sigma = (\sigma_0, \sigma_1, \dots)$, called "strategy" in what follows, where $\sigma_0(\cdot) = P(X_1 \in \cdot)$ is the marginal distribution of X_1 and $\sigma_n(\cdot) = P(X_{n+1} \in \cdot \mid X_1, \dots, X_n)$ the nth predictive distribution. Because of the Ionescu–Tulcea theorem, σ can be assigned directly, without passing through the usual prior/posterior scheme. One main advantage is that no prior probability is to be selected. In a nutshell, this is the predictive approach to Bayesian learning. A concise review of the latter is provided in this paper. We try to put such an approach in the right framework, to make clear a few misunderstandings, and to provide a unifying view. Some recent results are discussed as well. In addition, some new strategies are introduced and the corresponding distribution of the data sequence X is determined. The strategies concern generalized Pólya urns, random change points, covariates and stationary sequences.

Key words and phrases: Bayesian inference, conditional identity in distribution, exchangeability, predictive distribution, sequential predictions, stationarity.

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Exchangeability, Prediction and Predictive Modeling in Bayesian Statistics

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Abstract. There is currently a renewed interest in the Bayesian *predictive* approach to statistics. This paper offers a review on foundational concepts and focuses on "predictive modeling," which by directly reasoning on prediction bypasses inferential models or may characterize them. We detail predictive characterizations in exchangeable and partially exchangeable settings, for a large variety of data structures, and hint at new directions. The underlying concept is that Bayesian predictive rules are probabilistic *learning* rules, formalizing through conditional probability how we learn on future events given the available information. This concept has implications in any statistical problem and in inference, from classic contexts to less explored challenges, such as providing Bayesian uncertainty quantification to predictive algorithms in data science, as we show in the last part of the paper. The paper not only gives a historical overview but also includes a few new results, presents some recent developments and poses some open questions.

Key words and phrases: Bayesian foundations, predictive characterizations, Bayesian nonparametrics, predictive sufficiency, partial exchangeability, recursive algorithms.

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Martingale Posterior Distributions for Time-Series Models

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Abstract. The paper considers the analysis of time-series data from a future missing data perspective. Using predictive models with random multistep, updating of the parameters is equivalent to an iterative ensemble approach where initial conditions are randomized. The output is a sample from a posterior distribution equivalent in structure to the martingale posterior distribution. Specifically, the paper provides a posterior distribution of the parameters of a time-series model by forward sampling from the data using a predictive model based on iteratively sampling and updating the model until the sequence of parameter estimators has converged. Repetition of this produces independent samples from the posterior. Illustrative and real data examples are provided.

Key words and phrases: Martingale, missing future data, multistep forecasting, score function.

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Bayesian Dependent Mixture Models: A Predictive Comparison and Survey

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Abstract. For exchangeable data, mixture models are an extremely useful tool for density estimation due to their attractive balance between smoothness and flexibility. When additional covariate information is present, mixture models can be extended for flexible regression by modeling the mixture parameters, namely, the weights and atoms, as functions of the covariates. These types of models are interpretable and highly flexible, allowing not only the mean but the whole density of the response to change with the covariates. This article reviews Bayesian covariate-dependent mixture models, focusing on nonparametric constructions which we broadly organize into into three categories: (1) joint models of the responses and covariates, (2) conditional models with single-weights and covariate-dependent atoms, and (3) conditional models with covariate-dependent weights. The diversity and variety of the available models in the literature raises the question of how to choose among them for the application at hand. We attempt to shed light on this question through a careful analysis of the predictive equations for the conditional mean and density function as well as predictive comparisons in a variety of simulated data examples.

Key words and phrases: Density regression, dependent Dirichlet process, mixture of experts, nonparametric regression, stick-breaking representation.

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Bayesian Predictive Synthesis with Outcome-Dependent Pools

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Abstract. This paper reviews background and examples of Bayesian predictive synthesis (BPS), and develops details in a subset of BPS mixture models. BPS expands on standard Bayesian model uncertainty analysis for model mixing to provide a broader foundation for calibrating and combining predictive densities from multiple models or other sources. One main focus here is BPS as a framework for justifying and understanding generalized "linear opinion pools," where multiple predictive densities are combined with flexible mixing weights that depend on the forecast variable itself– that is, the setting of outcome-dependent model mixing. BPS also defines approaches to incorporating and exploiting dependencies across models defining forecasts, and to formally addressing the problem of model set incompleteness within the subjective Bayesian framework. In addition to an overview of general mixture-based BPS, new methodological developments for dynamic BPS—involving calibration and pooling of sets of predictive distributions in a univariate time series setting—are presented. These developments are exemplified in summaries of an analysis in a univariate financial time series study.

Key words and phrases: Bayesian predictive synthesis, density forecast combination, forecaster dependence, forecasting, forecast calibration, generalized opinion pools, model combination, model set incompleteness, time series prediction.

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Advances in Projection Predictive Inference

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Abstract. The concepts of Bayesian prediction, model comparison, and model selection have developed significantly over the last decade. As a result, the Bayesian community has witnessed a rapid growth in theoretical and applied contributions to building and selecting predictive models. Projection predictive inference in particular has shown promise to this end, finding application across a broad range of fields. It is less prone to over-fitting than naïve selection based purely on cross-validation or information criteria performance metrics, and has been known to out-perform other methods in terms of predictive performance. We survey the core concept and contemporary contributions to projection predictive inference, and present a safe, efficient, and modular workflow for prediction-oriented model selection therein. We also provide an interpretation of the projected posteriors achieved by projection predictive inference in terms of their limitations in causal settings.

Key words and phrases: Bayesian model selection, cross-validation, projection predictive inference.

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A Conversation with A. Philip Dawid

Vladimir Vovk and Glenn Shafer

Abstract. Beginning in the 1970s, Alexander Philip Dawid has been a leading contributor to the foundations of statistics and especially to the development and application of Bayesian statistics. He is also known for his work on causality, especially his notation for conditional independence and his critique of the overuse of counterfactuals, and for his contributions to forensic statistics.

Dawid was born in Lancashire, England, on February 1, 1946. His family moved to London soon afterwards, and he attended the City of London School from 1956 to 1963. He studied mathematics at Cambridge, earning a BA (Bachelor of Arts) degree in 1966. After earning a Diploma in Mathematical Statistics in the academic year 1966–1967, he studied for a PhD at Imperial, then at UCL, where he became a Lecturer in Statistics in 1969. In 1978, he left UCL for a position as Professor of Statistics in the Department of Mathematics, The City University, London, where he served as Head of Statistics Section and Director of the Statistical Laboratory. He returned to the Department of Statistics at UCL in 1981, serving as Head of Department from 1983 to 1993. He moved to the University of Cambridge in 2007, becoming Professor of Statistics and Fellow of Darwin College. He has continued his work in mathematical statistics after retiring from Cambridge in 2013 and was elected Fellow of the Royal Society in 2018.

Key words and phrases: A. Philip Dawid, biography, Bayesian statistics, prequential statistics, causality, University College London.

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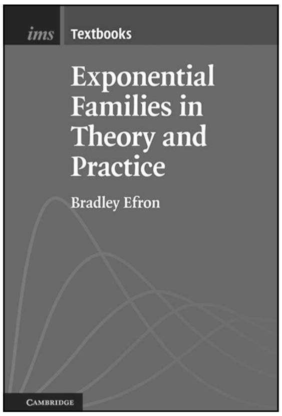

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