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Introduction to the Special Issue on Reinforcement Learning

Nan Jiang and Ambuj Tewari

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Sample-Based Planning and Learning with Function Approximation

Tor Lattimore and Csaba Szepesvári

Abstract. We give a short tutorial on sample-based planning and learning with function approximation for reinforcement learning. The focus is on explaining the standard setups and algorithm design principles. The core algorithmic ideas explained are approximate backwards induction, the (Bellman) eluder dimension and the optimism principle.

Key words and phrases: Reinforcement learning, planning.

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On the Statistical Complexity for Offline and Low-Adaptive Reinforcement Learning with Structures

Ming Yin, Mengdi Wang and Yu-Xiang Wang

Abstract. This article reviews the recent advances on the statistical foundation of reinforcement learning (RL) in the offline and low-adaptive settings. We will start by arguing why offline RL is the appropriate model for almost any real-life ML problems, even if they have nothing to do with the recent AI breakthroughs that use RL. Then we will zoom into two fundamental problems of offline RL: offline policy evaluation (OPE) and offline policy learning (OPL). It may be surprising to people that tight bounds for these problems were not known even for tabular and linear cases until recently. We delineate the differences between worst-case minimax bounds and instance-dependent bounds. We also cover key algorithmic ideas and proof techniques behind near-optimal instance-dependent methods in OPE and OPL. Finally, we discuss the limitations of offline RL and review a burgeoning problem of *low-adaptive exploration* which addresses these limitations by providing a sweet middle ground between offline and online RL.

Key words and phrases: Sample complexity, offline reinforcement learning, low-adaptive exploration.

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Offline Reinforcement Learning in Large State Spaces: Algorithms and Guarantees

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Abstract. This article introduces the theory of offline reinforcement learning in large state spaces, where good policies are learned from historical data without online interactions with the environment. Key concepts introduced include expressivity assumptions on function approximation (e.g., Bellman-completeness vs. realizability) and data coverage (e.g., all-policy vs. single-policy coverage). A rich landscape of algorithms and results is described, depending on the assumptions one is willing to make and the sample and computational complexity guarantees one wishes to achieve. We also discuss open questions and connections to adjacent areas.

Key words and phrases: Offline reinforcement learning, function approximation.

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The Central Role of the Loss Function in Reinforcement Learning

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Abstract. This paper illustrates the central role of loss functions in data-driven decision making, providing a comprehensive survey on their influence in cost-sensitive classification (CSC) and reinforcement learning (RL). We demonstrate how different regression loss functions affect the sample efficiency and adaptivity of value-based, decision-making algorithms. Across multiple settings, we prove that algorithms using the binary cross-entropy loss achieve first-order bounds scaling with the optimal policy's cost and are much more efficient than the commonly used squared loss. Moreover, we prove that distributional algorithms using the maximum likelihood loss achieve second-order bounds scaling with the policy variance and are even sharper than first-order bounds. This in particular proves the benefits of distributional RL. We hope that this paper serves as a guide analyzing decision-making algorithms with varying loss functions, and can inspire the reader to seek out better loss functions to improve any decision-making algorithm.

Key words and phrases: First-order (small-loss) and second-order (variance-dependent) bounds, RL with function approximation, distributional RL.

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Partially Observable RL: Benign Structures and Simple Generic Algorithms

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Abstract. Partially observable Reinforcement Learning (RL) applications, where agents must make a series of decisions without complete knowledge of the underlying states of the system, are widespread. Partially observable RL can be notoriously difficult—well-known information-theoretic results show that learning partially observable Markov decision processes (POMDPs) requires an exponential number of samples in the worst case. However, this does not preclude the possibility of identifying rich subclasses of structured POMDPs for which learning remains feasible.

This survey aims to offer a high-level overview of recent advancements in learning structured POMDPs. We will identify clean and practical problem structures that facilitate sample-efficient learning. Additionally, we will introduce simple and generic algorithms for learning POMDPs under different structural conditions. Finally, we will provide a unified view of these results under the framework of well-conditioned predictive state representation, which further reveals new tractable classes of partially observable problems along the way.

Key words and phrases: Partially observable reinforcement learning.

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The Theory of Online Control

Elad Hazan and Karan Singh

Abstract. This article introduces the reader to the fundamentals of a novel algorithmic paradigm in control of dynamical systems called *online control*. The new approach is closely related to the decision-making framework of online convex optimization. This methodology is applied together with convex relaxations to obtain new methods that yield provable guarantees even in the absence of distributional assumptions.

The differences between online control and other control frameworks stem from its unorthodox objective. In optimal control, robust control, and other control methodologies that assume stochastic noise, the goal is to design controllers that perform as well as an *ex ante* optimal strategy. In online control, both the cost functions as well as the perturbations to the dynamical model are chosen online by an adversary and, therefore, the optimal policy is not defined a priori. Instead, the objective of an online controller is to attain low regret against the best policy chosen in hindsight from a benchmark class of policies.

To achieve this objective, online control algorithms adapt the decision making framework of online convex optimization to stateful systems, and are based on iterative mathematical optimization algorithms. These algorithms confer finite-time regret bounds and computational complexity guarantees. After an introduction to the core algorithms and principles, we survey recent advances and extensions of this methodology to time-varying systems, deep neural controllers, partial observation, system identification, planning and model-free reinforcement learning.

Key words and phrases: Nonstochastic control, online learning, LQR.

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Stochastic Approximation and Reinforcement Learning: The Interface and a Little Beyond

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Abstract. This article traces the coevolution of stochastic approximation and reinforcement learning. Beginning with a brief historical background on both, it will highlight the points of intersection and the spin-offs thereof, summarizing results in reinforcement learning that have benefited from stochastic approximation theory. The article ends by outlining further developments and challenges that remain.

Key words and phrases: Stochastic approximation, reinforcement learning, almost supermartingales, "ODE" approach, convergence.

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Replicable Bandits for Digital Health Interventions

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Abstract. Adaptive treatment assignment algorithms, such as bandit algorithms, are increasingly used in digital health intervention clinical trials. Frequently, the data collected from these trials is used to conduct causal inference and related data analyses to decide how to refine the intervention and whether to roll out the intervention more broadly. This work studies inference for estimands that depend on the adaptive algorithm itself; a simple example is the mean reward under the adaptive algorithm. Specifically, we investigate the replicability of statistical analyses concerning such estimands when using data from trials deploying adaptive treatment assignment algorithms. We demonstrate that many standard statistical estimators can be inconsistent and fail to be replicable across repetitions of the clinical trial, even as the sample size grows large. We show that this nonreplicability is intimately related to properties of the adaptive algorithm itself. We introduce a formal definition of a "replicable bandit algorithm" and prove that under such algorithms, a wide variety of common statistical estimators are guaranteed to be consistent and asymptotically normal. We present both theoretical results and simulation studies based on a mobile health oral health self-care intervention. Our findings underscore the importance of designing adaptive algorithms with replicability in mind, especially for settings like digital health, where deployment decisions rely heavily on replicated evidence. We conclude by discussing open questions on the connections between algorithm design, statistical inference, and experimental replicability.

Key words and phrases: bandit algorithms, digital health, replicability, adaptive treatment assignment.

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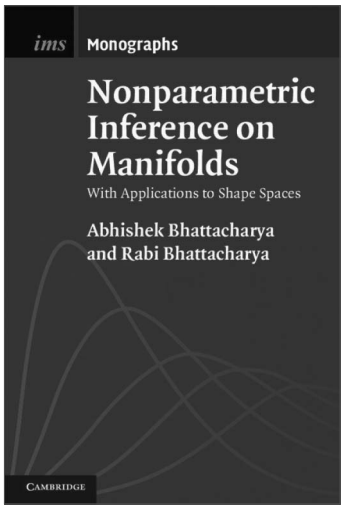

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