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Bayesian Nonparametric Inference for "Species-Sampling" Problems

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Abstract. Given an observed sample from a population of individuals belonging to species, "species-sampling" problems (SSPs) call for estimating some features of the unknown species composition of additional unobservable samples from the same population. Within SSPs the problems of estimating coverage probabilities, the number of unseen species and coverages of prevalences have emerged in the past three decades for being the subject of numerous methodological and applied works, mostly in biological sciences but also in statistical machine learning, electrical engineering, theoretical computer science, information theory and forensic statistics. In this paper we focus on these popular SSPs and present an overview of their Bayesian nonparametric (BNP) analysis under the Pitman–Yor process (PYP) prior. While reviewing the literature, we improve on computation and interpretability of existing posterior inferences, typically expressed through complicated combinatorial numbers, by establishing novel posterior representations in terms of simple compound binomial and hypergeometric distributions. We also consider the problem of estimating the discount and scale parameters of the PYP prior, showing a property of Bayesian consistency with respect to estimation through the hierarchical Bayes and empirical Bayes approaches; that is, the discount parameter can be estimated consistently, whereas the scale parameter cannot be estimated consistently, thus advising caution in posterior inference. We conclude our work by discussing some generalizations of SSPs, mostly in the field of biological sciences, which deal with "feature-sampling," multiple populations of individuals sharing species and classes of Markov chains.

Key words and phrases: Bayesian nonparametrics, Bayesian consistency, coverage of prevalences, coverage probabilities, empirical Bayes, hierarchical Bayes, Pitman–Yor process prior, "species-sampling" problems, unseen species.

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On Matérn Covariance and Gaussian Markov Random Fields: A Spectral Analysis

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Abstract. That Gaussian Markov random fields can be used to approximate Gaussian random fields with Matérn covariances functions gained widespread attention due to the seminal work of Lindgren, Rue, and Lindström (*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **73** (2011) 423–498). This was the culmination of a rich history within the statistical literature with pivotal contributions by Whittle (*Biometrika* **41** (1954) 434–449), Moran (*J. Appl. Probab.* **10** (1973) 54–62), and Besag (*J. Roy. Statist. Soc. Ser. B* (1974) 192–236). There developments often relied on analogies to time series analysis where the Markov properties of autoregressive processes were well understood. We revisit this literature while simultaneously offering an alternative demonstration of the result by Lindgren, Rue, and Lindström (*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **73** (2011) 423–498) that also relies on time series techniques, namely, spectral analysis. New insight is gained into understanding the role of the curse of dimensionality: as dimension increases, neighbourhoods of increasing size are required in order to exploit the GMRF-GRF approximation. Here the rigour of the differential equation argument in Lindgren, Rue, and Lindström (*J. R. Stat. Soc. Ser. B. Stat. Methodol.* **73** (2011) 423–498) is sacrificed for the sake of clarity and simplicity. The result is accessible to students enrolled in their first course in time series analysis.

Key words and phrases: Autoregression, curse of dimensionality, frequency domain, lattice processes, spatial statistics, spectral analysis.

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Overlap Measures Against ROC Summary Indices

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Abstract. Several indices of accuracy have been proposed to summarize ROC curves. Based on these indices, statistical tests have been developed to compare the accuracy of diagnostic tests. We briefly review three of them, *AUC*, Youden index, and the length of the ROC curve. Besides, we explore overlap measures as alternative measures for assessing the effectiveness of a diagnostic marker. Both parametric and nonparametric estimators as well as different methods for constructing confidence intervals are proposed for these measures. We show that they are related to distances and divergences and provide some properties. We then identify situations in which the overlap measures outperform the ROC summary indices through a simulation study. Furthermore, we compare the methods for constructing confidence intervals in terms of coverage and width. Our approaches are illustrated using a real data set.

Key words and phrases: ROC curve, *AUC*, Youden index, length of ROC curve, Weitzman measure, Kullback–Leibler measure, Matusita measure, bootstrap, kernel.

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Evidence Bounds in Singular Models: Probabilistic and Variational Perspectives

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Abstract. In Bayesian statistics, the marginal likelihood, a.k.a. the evidence, contains an intrinsic penalty accounting for larger model sizes and is of fundamental importance in Bayesian model comparison. Over the past two decades, there has been steadily increasing activity to understand the nature of this penalty in singular statistical models, building on pioneering works by Sumio Watanabe. Unlike regular models where the Bayesian information criterion (BIC) encapsulates a first-order expansion of the evidence, parameter counting gets trickier in singular models where a quantity called the real log-canonical threshold (RLCT) summarizes the effective model dimensionality. In this article, we offer a probabilistic treatment to recover nonasymptotic versions of established evidence bounds as well as prove a new result based on the Gibbs variational inequality. In particular, we show that mean-field variational inference correctly recovers the RLCT for any singular model in its standard form. We additionally exhibit sharpness of our bound empirically in dimension $d = 2$ and provide two conjectures concerning the asymptotics of the mean-field ELBO for singular models in standard form.

Key words and phrases: Bayesian model selection, coordinate ascent, Gibbs variational inequality, Laplace approximation, mean-field approximation, real log-canonical threshold.

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Doubly Ranked Tests of Location for Grouped Functional Data

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Abstract. Nonparametric tests for functional data are a challenging class of tests to work with because of the potentially high-dimensional nature of the data. One of the main challenges for considering rank-based tests, like the Mann–Whitney or Wilcoxon Rank Sum tests (MWW), is that the unit of observation is typically a curve. Thus, any rank-based test must consider ways of ranking curves. While several procedures, including depth-based methods, have recently been used to create scores for rank-based tests, these scores are not constructed under the null and often introduce additional, uncontrolled for variability. We therefore reconsider the problem of rank-based tests for functional data and develop an alternative approach that incorporates the null hypothesis throughout. Our approach first ranks realizations from the curves at each measurement occurrence, then calculates a summary statistic for the ranks of each subject, and finally reranks the summary statistic in a procedure we refer to as a doubly ranked test. We propose two summaries for the middle step: a sufficient statistic and the average rank. As we demonstrate, doubly rank tests are more powerful while maintaining ideal type I error in the two sample, MWW setting. We also extend our framework to more than two samples, developing a Kruskal–Wallis test for functional data, which exhibits good test characteristics as well. Finally, we illustrate the use of doubly ranked tests in functional data contexts from material science, climatology and public health policy.

Key words and phrases: Nonparametric functional data analysis, Wilcoxon rank sum, Mann–Whitney test, Kruskal–Wallis test, order statistics, sufficient statistics.

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A Primer on Bayesian Neural Networks: Review and Debates

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Abstract. Neural networks have achieved remarkable performance across various problem domains, but their widespread applicability is hindered by inherent limitations such as overconfidence in predictions, lack of interpretability and vulnerability to adversarial attacks. To address these challenges, Bayesian neural networks (BNNs) have emerged as a compelling extension of conventional neural networks, integrating uncertainty estimation into their predictive capabilities.

This comprehensive primer presents a systematic introduction to the fundamental concepts of neural networks and Bayesian inference, elucidating their synergistic integration for the development of BNNs. The target audience comprises statisticians with a potential background in Bayesian methods but lacking deep learning expertise, as well as machine learners proficient in deep neural networks but with limited exposure to Bayesian statistics. We provide an overview of commonly employed priors, examining their impact on model behavior and performance. Additionally, we delve into the practical considerations associated with training and inference in BNNs.

Furthermore, we explore advanced topics within the realm of BNN research, acknowledging the existence of ongoing debates and controversies. By offering insights into cutting-edge developments, this primer not only equips researchers and practitioners with a solid foundation in BNNs, but also illuminates the potential applications of this dynamic field. As a valuable resource, it fosters an understanding of BNNs and their promising prospects, facilitating further advancements in the pursuit of knowledge and innovation.

Key words and phrases: Approximate Bayesian inference, deep learning, neural networks.

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An Overview of Asymptotic Normality in Stochastic Blockmodels: Cluster Analysis and Inference

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Abstract. This paper provides a selective review of the statistical network analysis literature focused on clustering and inference problems for stochastic blockmodels and their variants. We survey asymptotic normality results for stochastic blockmodels as a means of thematically linking classical statistical concepts to contemporary research in network data analysis. Multiple different forms of asymptotically Gaussian behavior arise in stochastic blockmodels and are useful for different purposes, pertaining to estimation and testing, the characterization of cluster structure in community detection and understanding latent space geometry. This paper concludes with a discussion of open problems and ongoing research activities addressing asymptotic normality and its implications for statistical network modeling.

Key words and phrases: Random graphs, statistical network analysis, community detection, clustering, limit theorems.

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Approximate Inference with Exponential Tilting Densities: Theory and Applications

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Abstract. A family of exponential tilting density functions (ETD) is presented and compared with energy functions. This ETD family is shown to be associated with the normal density and the log-gamma density by the minimum cross entropy in information theory. In this paper, we show that ETD minimizes the Kulback–Leiber divergence under some moments constraints. Two examples are provided to illustrate how to approximate a baseline density using ETD. In addition, the normalizing constant of the ETD is approximated by three commonly used approximation methods: Gaussian variational approximation (GVA), Laplace approximation (LA) and saddlepoint approximation (SA). It is shown that the normalizing constant obtained by GVA and SA are asymptotically equivalent, and theoretically, both are more accurate than the one obtained by LA. With the availability of the normalizing constant, likelihood-based asymptotic inference can be obtained. To demonstrate the applicability of the proposed method, it is applied to parameter estimation of the Poisson mixed model and Bayesian inference.

Key words and phrases: Kullback–Leiber divergence, Laplace approximation, saddelpoint approximation, Gaussian variational approximation, Poisson mixed model, Bayesian inference, Poisson log-normal distribution.

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A Conversation with Byron J.T. Morgan

Rachel McCrea

Abstract. Byron Morgan is an Emeritus Professor of Statistics, a Fellow of the Learned Society of Wales and represents a true pillar of the applied statistics community, contributing appreciable understanding to a number of fields, most recently statistical ecology.

Byron has offered sustained service to the community, chairing the last Research Excellence Framework statistics panel in 2001 (after which statistics was merged with mathematics) and has reviewed and advised several external agencies at home and abroad. He has been editor of the Journal of Agricultural, Biological and Environmental Statistics, co-editor of Biometrics and of Applied Statistics and Associate Editor and Guest Editor for several other journals. He also served in multiple learned societies, most notably the International Biometric Society, (of which he was the president in 1996/1997 and Elected Honorary Life Member in 2014) and in the Royal Statistical Society, where he was a member of council and a vice-president during 1997–2001. His contributions to the organisation of scientific conferences are many and include his serving as Chair of Scientific Program Committee of the XVIIth International Biometric Conference in Hamilton, Ontario, 1994, and as Chair of the local organising committee for the International Statistical Ecology Conference, held in Canterbury in 2010.

As well as leading many initiatives and grants, he has mentored and inspired generations of researchers, many of whom have gone on to research careers in both academia and NGOs. Anyone who has met Byron will not be surprised that it is a great pleasure to have an excuse to talk to him about his career and for him to relay some of his many entertaining stories.

Key words and phrases: Capture-recapture, integrated population modelling, National Centre for Statistical Ecology, International Statistical Ecology Conference, parameter redundancy, population dynamics.

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Re-Thinking Spatial Confounding in Spatial Linear Mixed Models

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Abstract. In the last two decades, considerable research has been devoted to a phenomenon known as spatial confounding. Spatial confounding is thought to occur when there is multicollinearity between a covariate and the random effect in a spatial regression model. This multicollinearity is considered highly problematic when the inferential goal is estimating regression coefficients and various methodologies have been proposed to attempt to alleviate it. Recently, it has become apparent that many of these methodologies are flawed, yet the field continues to expand. In this paper, we offer a novel perspective of synthesizing the work in the field of spatial confounding. We propose that at least two distinct phenomena are currently conflated with the term spatial confounding. We refer to these as the "analysis model" and the "data generation" types of spatial confounding. We show that these two issues can lead to contradicting conclusions about whether spatial confounding exists and whether methods to alleviate it will improve inference. Our results also illustrate that in most cases, traditional spatial linear mixed models *do* help to improve inference on regression coefficients. Drawing on the insights gained, we offer a path forward for research in spatial confounding.

Key words and phrases: Spatial regression, geostatistics, areal data, omitted variable bias.

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The Intersection of Location-Allocation, Partitional Clustering and Model-Based Clustering Techniques—a Review

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Abstract. Location-allocation, partitional spatial clustering and model-based clustering all deal with spatial data, seemingly from different viewpoints. Partitional clustering analyzes data points by partitioning them into separate groups, location-allocation places facilities in locations that best meet the needs of demand points, while model-based clustering provides a probabilistic interpretation of the underlying clusters. However, both partitional clustering and location-allocation can be formulated as optimization problems minimizing the distances of (demand) points from their associated centers (facilities). This common framework further connects to model-based clustering by means of likelihood maximization. Moreover, all of these approaches consider extensions such as weighted data points, different distance metrics, capacity constraints, different membership types, outliers and selecting the number of clusters or facilities.

In this article, we highlight and review the similarities and differences of these techniques. We look at a number of extensions common for all approaches, and consider their treatment under the framework. Finally, we provide a number of detailed examples highlighting the effects of extensions.

Key words and phrases: Clustering, spatial clustering, location-allocation, location analytics, finite mixture model.

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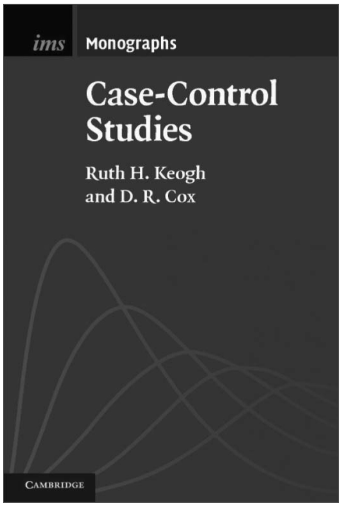

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Ruth H. Keogh
and D. R. Cox

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