Estimation of parameters in the DDRCINAR(p) model

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Abstract. This paper discusses a pth-order dependence-driven random coefficient integer-valued autoregressive time series model (DDRCINAR(p)). Stationarity and ergodicity properties are proved. Conditional least squares, weighted least squares and maximum quasi-likelihood are used to estimate the model parameters. Asymptotic properties of the estimators are presented. The performances of these estimators are investigated and compared via simulations. In certain regions of the parameter space, simulative analysis shows that maximum quasi-likelihood estimators perform better than the estimators of conditional least squares and weighted least squares in terms of the proportion of within-Ω estimates. At last, the model is applied to two real data sets.

1 Introduction

Integer-valued time series models have received growing attention recently. These models can be broadly classified into two types. One is regression-type models, and the other is ‘thinning’ models. (See Davis, R. A., Dunsmuir, W. T. M. and Wang, Y. (1999), for a review of the regression models.) Steutel, F. W. and Van Harn, K. (1979) gave the ‘thinning’ operator ‘◦’, which has been used by many authors. For example, Al-Osh, M. A. and Alzaid, A. A. (1987), Alzaid, A. and Al-Osh, M. (1988) and Alzaid, A. A. and Al-Osh, M. (1990) have studied the thinning models, as well as Jin-Guan, D. and Yuan, L. (1991), Latour, A. (1997, 1998), Brannas, K. and Hellstrom, J. (2001) and Li, C., Wang, D. and Zhang, H. (2015), in a slightly more general form, among others. The work presented in this paper is that we propose a dependence-driven random coefficient thinning model for the pth-order integer-valued autoregression.

The first-order integer-valued autoregressive (INAR(1)) process is introduced by Al-Osh, M. A. and Alzaid, A. A. (1987). It is defined by

$$X_t = \phi \circ X_{t-1} + \varepsilon_t, \quad t \geq 1,$$

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where

\[ \phi \circ X_{t-1} = \sum_{i=1}^{X_{t-1}} B_{i,t}, \]

here, the so-called counting series \( \{B_{1,t}\} \) are independent and identically distributed (i.i.d.) Bernoulli random variables with success probability \( \phi \in [0,1] \) and \( \{\varepsilon_t\} \) is a sequence of i.i.d. non-negative integer-valued random variables and independent of the counting series \( \{B_{1,t}\} \). Thus, \( \phi \circ X_{t-1} \) is a binomial random variable with \( \phi \) and \( X_{t-1} \) as parameters, namely \( \phi \circ X_{t-1} \sim B(X_{t-1}, \phi) \).

In many real-life situations there is a INAR model to be used for some situations, which could be found in reliability theory, meteorology, insurance theory, communications, medicine, law and social sciences, such as counts of accidents, detected errors, transmitted messages, patients, crime victimization, etc. As an example of a standard INAR(1) model, let \( X_t \) denote the number of surviving epileptic patients in a hospital at time \( t \), \( \phi \) the probability of survival from time \( t-1 \) to \( t \), and \( \varepsilon_t \) the number of new epileptic patients admitted at time \( t \). As further example, suppose \( X_t \) denote the number of unemployed in the \( t \)th month. Then \( X_t \) can be modeled as the sum of the previously unemployed \( \phi \circ X_{t-1} \) and the newly unemployed \( \varepsilon_t \). In particular, the parameter \( \phi \) may vary with time and it may be random as the survival rate (the unemployment rate) \( \phi \) may be affected by various environmental factors, such as the quality of health care, the state of health of patients, etc (affected by factors such as the state of the economy, productivity growth, etc). Thus, it is necessary to research random-coefficient INAR model. Recently, Zheng, H., Basawa, I. V. and Datta, S. (2007) and Zheng, H. and Basawa, I. V. (2008) studied the random-coefficient INAR(1) model as well as Zhang, H. and Wang, D. (2015).

The \( p \)th-order integer-valued autoregressive (INAR(\( p \)) model is recursively defined by Jin-Guan, D. and Yuan, L. (1991) as

\[ X_t = \sum_{i=1}^{p} \phi_i \circ X_{t-i} + \varepsilon_t, \quad t \geq 1, \]

where, for \( i = 1, \cdots, p \),

\[ \phi_i \circ X_{t-i} = \sum_{i=1}^{X_{t-i}} B_{i,t}, \]

here \( \{B_{i,t}\}, i \in \{1, \cdots, p\} \) are independent Bernoulli-distributed variables, where \( \{B_{i,t}\} \) has success probability \( \phi_i \in [0,1] \) and \( \{\varepsilon_t\} \) is a sequence of
i.i.d. non-negative integer-valued random variables and independent of all the counting series and \( \sum_{i=1}^{p} \phi_i < 1 \).


\[
X_t = \sum_{i=1}^{p} \phi_i^{(t)} \circ X_{t-i} + \varepsilon_t, \quad t \geq 1,
\]

where the random parameter \( \{\phi_i^{(t)}\} \) replaces the fixed \( \{\phi_i\} \) values in the literature by Jin-Guan, D. and Yuan, L. (1991), and is an i.i.d. sequence for fixed \( i \) and \( \sum_{i=1}^{p} E(\phi_i^{(t)}) < 1 \). However, in the practical-life situations, \( \{\phi_i^{(t)}\} \) may be dependent in some kind of relationship. In this paper, we extend the above model to a dependence-driven random coefficient model DDRCINAR(p), where \( \{\phi_i\} \) is a dependence-driven sequence of random vectors with a joint distribution function \( P_{\{\phi_{t1}, \ldots, \phi_{tp}\}} \). Therefore, this article is mainly to introduce the basic statistical properties of this model and provide some inferential methods for the relevant parameters associated with this model.

The structure of the article is as follows. In Section 2, the dependence-driven random coefficient model DDRCINAR(p) is described in detail and we show, under certain conditions, the stationarity and the ergodicity of the DDRCINAR(p) model. In Section 3, we propose three estimation methods for the DDRCINAR(p) model parameters and study their consistency and asymptotic properties. In Section 4, comparisons among the three methods for the DDRCINAR(2) model and the proportion of in-range estimates are given via simulation studies. In Section 5, two real data sets are analysed by using estimation methods in Section 3. In Section 6, we give a summary and concluding remarks. The paper ends with conditional moments used later which are provided in the Appendix.
2 The \( p \)Th-order Dependence-driven Random Coefficient Integer-valued Autoregressive Model

A \( p \)th-order dependence-driven random coefficient integer-valued autoregressive (DDRCINAR(\( p \))) model is defined by the following equation:

\[
X_t = \sum_{i=1}^{p} \phi_{ti} \circ X_{t-i} + \varepsilon_t, \quad t \geq 1, \tag{2.1}
\]

where \( \{ \varepsilon_t \} \) is an i.i.d. non-negative integer-valued sequence with a probability mass function \( f_{\varepsilon_t} > 0 \), such that \( E(\varepsilon_t^4) < \infty \); \( \{ \phi_{ti}, 1 \leq i \leq p \} \) and \( \{ \varepsilon_t \} \) are independent each other; the joint distribution of \( \{ \phi_{t1}, \phi_{t2}, \ldots, \phi_{tp} \} \) is given by

\[
\begin{align*}
&\begin{cases}
p(\phi_{t1} = \phi_1, \phi_{t2} = 0, \ldots, \phi_{tp} = 0) = \alpha_1; \\
p(\phi_{t1} = 0, \phi_{t2} = \phi_2, \ldots, \phi_{tp} = 0) = \alpha_2; \\
\vdots \\
p(\phi_{t1} = 0, \phi_{t2} = 0, \ldots, \phi_{tp} = \phi_p) = \alpha_p; \\
p(\phi_{t1} = 0, \phi_{t2} = 0, \ldots, \phi_{tp} = 0) = \alpha_0,
\end{cases} \\
\end{align*}
\tag{2.2}
\]

where \( \alpha_0, \alpha_1, \ldots, \alpha_p \) are non-negative and \( \sum_{i=0}^{p} \alpha_i = 1 \). Let

\[
\mu_\varepsilon = E(\varepsilon_t), \quad \sigma_\varepsilon^2 = Var(\varepsilon_t).
\]

According to (2.2), we have

\[
E(\phi_{ti}) = \alpha_i \phi_i,
\]

\[
Cov(\phi_{ti}, \phi_{tj}) = -\alpha_i \alpha_j \phi_i \phi_j, \quad Var(\phi_{ti}) = \alpha_i \phi_i^2 (1 - \alpha_i). \tag{2.3}
\]

Now, we give some notation for the following theorem.

Let

\[
\begin{align*}
X_t &= (X_t, X_{t-1}, \ldots, X_{t-p+1})'_1 \times p, \\
\varepsilon_t &= (\varepsilon_t, 0, \ldots, 0)'_1 \times p \quad \text{and} \quad \mu_\varepsilon = (\mu_\varepsilon, 0, \ldots, 0)'_1 \times p.
\end{align*}
\]

Then we have

\[
X_t = A_t \circ X_{t-1} + \varepsilon_t,
\]

where

\[
A_t = \begin{bmatrix}
\phi_{t1} & \phi_{t2} & \cdots & \phi_{tp-1} & \phi_{tp} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}.
\]
Theorem 2.1. If \( \sum_{i=1}^{p} \alpha_i \phi_i < 1 \) and the maximum absolute eigenvalue of \( E[A_t^t \otimes A_t] \) is less than 1, then there exists a unique stationary integer-valued random series \( \{X_t\} \) satisfying equation (2.1). Furthermore, the process is an ergodic process.

Proof. Firstly, we introduce a sequence of random variables \( \{Z_t^{(n)}\}_{n \in \mathbb{N}} \),

\[
Z_t^{(n)} = \begin{cases} 
0, & n < 0, \\
\varepsilon_t, & n = 0, \\
\sum_{i=1}^{p} \phi_{ti} \circ Z_{t-i}^{(n-i)} + \varepsilon_t, & n > 0.
\end{cases}
\]

Let

\[
U(n, t, k) = |Z_t^{(n)} - Z_t^{(n-k)}| \quad \text{and} \quad l(n, t, k) = \min(Z_t^{(n)}, Z_t^{(n-k)}).
\]

Then we have the following inequality

\[
U(n, t, k) \leq \sum_{i=1}^{p} |\phi_{ti} \circ Z_{t-i}^{(n-i)} - \phi_{ti} \circ Z_{t-i}^{(n-i-k)}|
\]

\[
= \sum_{i=1}^{p} \left| \sum_{j=1}^{Z_{t-i}^{(n-i)}} B_j^{(t,i)} - \sum_{j=1}^{Z_{t-i}^{(n-i-k)}} B_j^{(t,i)} \right|
\]

\[
= \sum_{i=1}^{p} U(n-i, t-i, k) + B_{l(n,t,k)+j}^{(t,i)}
\]

Let

\[
B = \begin{bmatrix} 
\alpha_1 \phi_1 (1 - \phi_1) & \alpha_2 \phi_2 (1 - \phi_2) & \cdots & \alpha_p \phi_p (1 - \phi_p) \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix},
\]
\[
A_t \circ \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^p \phi_{ti} \circ X_i \\ X_1 \\ \vdots \\ X_{p-1} \end{bmatrix}.
\]

Now we note that
\[
U_{t,k}^{(n)} = (U(n, t, k), U(n-1, t-1, k), \ldots, U(n-p+1, t-p+1, k))'.
\]

Then by equation (2.4) and the above notation, following the similar argument in Latour, A. (1998) or Zheng, H., Basawa, I. V. and Datta, S. (2006), we can obtain
\[
E(U_{t,k}^{(n)}) \leq A^n E(U_{t-1,k}^{(n-1)}) = A^n \mu_{\varepsilon}
\]
and
\[
E(U_{t,k}^{(n)} U_{t,k}^{(n)'}) \leq \text{diag}(BA^{n-1}\mu_{\varepsilon}) + E(A_t E(U_{t-1,k}^{(n-1)} U_{t-1,k}^{(n)'}) A_t').
\]

Therefore
\[
\text{vec}(E(U_{t,k}^{(n)} U_{t,k}^{(n)'}) \leq \sum_{j=0}^{n-1} (E(A_t \otimes A_t))^j \text{vec}(\text{diag}(BA^{n-j-1}\mu_{\varepsilon}))
\]
\[
+ (E(A_t \otimes A_t))^n \text{vec}\left( \begin{bmatrix} \sigma^2_{\varepsilon} + \mu_{\varepsilon}^2 & 0_{p-1} \\ 0_{p-1} & 0_{p \times p-1} \end{bmatrix} \right) \tag{2.7}
\]

Because \(\{A_t\}\) is an i.i.d. random matrix sequence, all the above inequalities are elementwise.

Since maximum absolute eigenvalue of \(E(A_t' \otimes A_t)\) is less than 1, \([E(A_t' \otimes A_t)]^n\) converges to a null matrix. And \(A^n\) converges to a null matrix as well since \(\sum_{i=1}^p \alpha_i \phi_i < 1\). Thus, using a similar method as in Latour, A. (1998), we can prove that
\[
E(U_{t,k}^{(n)} U_{t,k}^{(n)'}) \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]

For the ergodicity of the process, we can follow the proof in Jin-Guan, D. and Yuan, L. (1991) based on Zikun, W. (1965). The only difference between Zheng, H., Basawa, I. V. and Datta, S. (2006) and Jin-Guan, D. and Yuan, L. (1991) is that Zheng, H., Basawa, I. V. and Datta, S. (2006) introduce that \( \{ \phi_1(t), \phi_2(t), \ldots, \phi_p(t) \}, \ t \geq 1 \) is an i.i.d. sequence with a cumulative distribution function \( P_{\phi} \) and assume \( E(\phi_i(t)) = \phi_i, (i = 1, 2, \ldots, p) \). But we assume that the random vectors \( \{ \phi_{t1}, \phi_{t2}, \ldots, \phi_{tp} \} \) are dependence-driven and have the joint distribution function as (2.2). Therefore, the argument on ergodicity in Jin-Guan, D. and Yuan, L. (1991) remains effective for our model. This condition, for the existence of the stationary and ergodic DDRCINAR(p) model, will define the parameter space \( \Omega \) used here for the model. We also show \( 0 \leq \alpha_i \leq 1, 0 \leq \phi_i \leq 1, i = 1, 2, \ldots, p \) and \( 0 < \alpha_1 + \alpha_2 + \cdots + \alpha_p < 1 \).

Next, we consider the problem of estimation involved in the DDRCINAR(p) model.

## 3 Estimation Methods

We re-parameterize equation (2.3) by defining

\[
a_i = \alpha_i \phi_i, \quad \sigma_{ii} = \alpha_i \phi_i^2 (1 - \alpha_i), \quad i = 1, 2, \ldots, p.
\]

(3.1)

Let \( \mathcal{F}_{t-1} \) be the \( \sigma \)-field generated by \( X_1, X_2, \ldots, X_{t-1} \), and \( \mathcal{F}_{tp} \) be the \( \sigma \)-field generated by \( \phi_{t1}, \phi_{t2}, \ldots, \phi_{tp} \). Denote

\[
a = (a_1, a_2, \ldots, a_p)', \quad \sigma = (\sigma_{11}, \sigma_{22}, \ldots, \sigma_{pp})', \quad \theta = (\alpha_1, \alpha_2, \ldots, \alpha_p, \phi_1, \phi_2, \ldots, \phi_p, \mu, \sigma^2)',
\]

Assume that observation of \( X_t \) are available for \( t = 1, 2, \ldots, n \).

Next, we consider three different methods of parameter estimation, namely, the conditional least squares (CLS) estimators, the weighted conditional least squares (WCLS) estimators and the maximum quasi-likelihood estimators (MQE). An advantage of the three methods is that they do not require specifying the exact family for the innovations.

### 3.1 Conditional Least Squares Estimators

The CLS estimates of \( a \) and \( \mu_\varepsilon \) can be obtained by minimizing

\[
Q_1(a, \mu_\varepsilon) = \sum_{t=p+1}^{n} u_t^2
\]
with respect to $a$ and $\mu_\varepsilon$, where $u_t = X_t - E(X_t|F_{t-1})$. This yields the estimators

$$
\hat{a} = \left( \sum_{t=p+1}^{n} Y_tY_t' - \frac{1}{n-p} \sum_{t=p+1}^{n} Y_t \sum_{t=p+1}^{n} Y_t' \right)^{-1} \times \left( \sum_{t=p+1}^{n} Y_tX_t - \frac{1}{n-p} \sum_{t=p+1}^{n} Y_t \sum_{t=p+1}^{n} X_t \right),
$$

(3.2)

$$
\hat{\mu}_\varepsilon = \frac{1}{n-p} \sum_{t=p+1}^{n} (X_t - Y_t'\hat{a}).
$$

(3.3)

with $Y_t = (X_{t-1}, X_{t-2}, \ldots, X_{t-p})'$.

To obtain estimates of $\sigma$ and $\sigma_\varepsilon^2$, CLS is again applied to estimate the residual sequence $H_t$, where

$$
\hat{H}_t = \left( X_t - \sum_{i=1}^{p} \hat{a}_i X_{t-i} - \mu_\varepsilon \right)^2 + 2 \sum_{j=2}^{p} \sum_{i=1}^{j-1} \hat{a}_i \hat{a}_j X_{t-i} X_{t-j} - \sum_{i=1}^{p} X_{t-i}(\hat{a}_i - \hat{a}_i^2),
$$

by minimizing

$$
Q_2(\sigma, \sigma_\varepsilon^2) = \sum_{t=p+1}^{n} \left( \hat{H}_t - \sum_{i=1}^{p} \hat{\sigma}_{ii}(X_{t-i}^2 - X_{t-i}) - \sigma_\varepsilon^2 \right)^2
$$

with respect to $\sigma$ and $\sigma_\varepsilon^2$. This yields the estimators

$$
\hat{\sigma} = \left( \sum_{t=p+1}^{n} Z_tZ_t' - \frac{1}{n-p} \sum_{t=p+1}^{n} Z_t \sum_{t=p+1}^{n} Z_t' \right)^{-1} \times \left( \sum_{t=p+1}^{n} Z_t\hat{H}_t - \frac{1}{n-p} \sum_{t=p+1}^{n} Z_t \sum_{t=p+1}^{n} \hat{H}_t \right),
$$

(3.4)

$$
\hat{\sigma}_\varepsilon^2 = \frac{1}{n-p} \sum_{t=p+1}^{n} (\hat{H}_t - Z_t'\hat{\sigma})
$$

(3.5)

with $Z_t = (X_{t-1}^2 - X_{t-1}, X_{t-2}^2 - X_{t-2}, \ldots, X_{t-p}^2 - X_{t-p})'$.

We obtain estimates $\hat{\vartheta}$ from

$$
\hat{\alpha}_i = \frac{\hat{a}_i^2}{\hat{\sigma}_{ii} + \hat{a}_i^2} \text{ and } \hat{\phi}_i = \frac{\hat{\sigma}_{ii} + \hat{a}_i^2}{\hat{a}_i}, \quad i = 1, 2, \ldots, p.
$$

(3.6)

The following theorem gives the strong consistency and the limited distribution of the estimates $\hat{\vartheta}$ given in equation (3.6).
Theorem 3.1. Let \( \{X_t\} \) be an DDRCINAR(p) process generated as in equation (2.1) and (2.2) with the conditions given in Theorem 2.1. Then the estimates \( \hat{\vartheta} \) obtained from equation (3.6) will be strongly consistent and jointly asymptotically normally distributed.

Proof. We first prove the strong consistency of \( \hat{a}, \hat{\sigma}, \hat{\mu}_\varepsilon \) and \( \hat{\sigma}_\varepsilon^2 \). According to Theorem 2.1, \( \{X_t\}_{t=1}^\infty \) is a stationary ergodic sequence of integrable random variables.

Let
\[
\begin{align*}
g_1(\theta^{(1)}, \mathcal{F}_{t-1}) &= E(X_t|\mathcal{F}_{t-1}) = \sum_{i=1}^{p} a_i X_{t-i} + \mu_\varepsilon, \\
Q_1(\theta^{(1)}) &= \sum_{t=p+1}^{n} (X_t - g_1(\theta^{(1)}, \mathcal{F}_{t-1}))^2,
\end{align*}
\]
then
\[
Q_1(\theta^{(1)}) = Q_1(\theta_0^{(1)}) + (\theta^{(1)} - \theta_0^{(1)})' \frac{\partial Q_1(\theta^{(1)})}{\partial \theta^{(1)}} + \frac{1}{2} (\theta^{(1)} - \theta_0^{(1)})' V_1 (\theta^{(1)} - \theta_0^{(1)}) + R_1,
\]
where \( \theta^{(1)} = (a', \mu_\varepsilon)' \). Take a Taylor expansion of \( Q_1(\theta^{(1)}) \) carried out to third order terms:
\[
Q_1(\theta^{(1)}) = Q_1(\theta_0^{(1)}) + (\theta^{(1)} - \theta_0^{(1)})' \frac{\partial Q_1(\theta^{(1)})}{\partial \theta^{(1)}} + \frac{1}{2} (\theta^{(1)} - \theta_0^{(1)})' V_1 (\theta^{(1)} - \theta_0^{(1)}) + R_1,
\]
where \( V_1 = \frac{\partial^2 Q_1(\theta_0^{(1)})}{\partial \theta^{(1)2}} \) and \( R_1 \) is the usual remainder term. Obviously, it is easy to check that \( g_1(\theta^{(1)}, \mathcal{F}_{t-1}) \), \( \frac{\partial g_1(\theta^{(1)}, \mathcal{F}_{t-1})}{\partial \theta^{(1)}} \), \( \frac{\partial^2 g_1(\theta^{(1)}, \mathcal{F}_{t-1})}{\partial \theta^{(1)2}} \) and \( \frac{\partial^3 g_1(\theta^{(1)}, \mathcal{F}_{t-1})}{\partial \theta^{(1)i} \partial \theta^{(1)j} \partial \theta^{(1)k}} \) for \( i, j, k \in \{1, 2, \cdots, p+1\} \) satisfy all the regularity conditions in Klimko, L. A. and Nelson, P. I. (1978). Thus, Theorem 3.1 of Klimko, L. A. and Nelson, P. I. (1978) leads us to conclude that \( \hat{\theta}^{(1)} \) is strongly consistent, which indicates that \( \hat{a} \) and \( \hat{\mu}_\varepsilon \) are strongly consistent.

Similarly, we obtain that \( \hat{\sigma} \) and \( \hat{\sigma}_\varepsilon^2 \) are strongly consistent. Then \( \hat{\vartheta} \) is strongly consistent from equation (3.6).

Next, we prove the asymptotic normality of the estimates \( \hat{\vartheta} \). According to theorem 3.1 of Hwang, S. Y. and Basawa, I. V. (1998) or theorem 3.1 of Nicholls, D. F. and Quinn, B. G. (2012), we have
\[
\sqrt{n} (\hat{a} - a) \overset{d}{\to} N_p(0, \Gamma^{-1} W \Gamma^{-1}), \quad n \to +\infty,
\]
where \( W = E(u_t^2 Y_t Y_t') = E(\text{var}(X_t|\mathcal{F}_{t-1})Y_t Y_t') \) and \( \Gamma = E(Y_t Y_t') \).

With the similar method, we can obtain
\[
\sqrt{n} (\hat{\sigma} - \sigma) \overset{d}{\to} N_p(0, L^{-1} \Sigma L^{-1}),
\]
\[ \sqrt{n}(\hat{\mu}_\varepsilon - \mu_\varepsilon) \xrightarrow{d} N(0, G), \]
\[ \sqrt{n}(\hat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2) \xrightarrow{d} N(0, I), \quad n \to +\infty. \]

According to theorem 3.2 in Nicholls, D. F. and Quinn, B. G. (2012), we have
\[ \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N_{2p+2}(0, \Theta), \quad n \to +\infty, \]
where
\[
\Theta = \begin{bmatrix}
\Gamma^{-1} W T^{-1} & \Gamma^{-1} V L^{-1} & \Gamma^{-1} M & \Gamma^{-1} D \\
L^{-1} V' T^{-1} & L^{-1} \Sigma L^{-1} & L^{-1} Q & L^{-1} B \\
M' T^{-1} & Q' L^{-1} & G & F \\
D' T^{-1} & B' L^{-1} & F & T
\end{bmatrix},
\]
where
\[
M = E(Y_t u_t^2), \quad V = E(u_t U_t Y_t Z_t') = E(\varphi_t Y_t Z_t'), \quad L = E(Z_t Z_t'),
\]
\[
G = E(u_t^2), \quad Q = E(Z_t U_t u_t), \quad \Sigma = E(U_t^2 Z_t Z_t'),
\]
\[
D = E(Y_t u_t U_t), \quad B = E(U_t^2 Z_t), \quad F = E(u_t U_t), \quad T = E(U_t^2),
\]
where
\[
\varphi_t = E(X_t^3|\mathcal{F}_{t-1}) - 3 \text{var}(X_t|\mathcal{F}_{t-1}) E(X_t|\mathcal{F}_{t-1}) - (E(X_t|\mathcal{F}_{t-1}))^3,
\]
\[
U_t = u_t^2 - E(u_t^2|\mathcal{F}_{t-1}).
\]

According to equation (3.6) and proposition 6.4.3 of Brockwell, P. J. and Davis, R. A. (2013), we have
\[ \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N_{2p+2}(0, \Phi \Theta \Phi'), \quad n \to +\infty, \]
where
\[
\Phi = \begin{bmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} \\
\Phi_{21} & \Phi_{22} & \Phi_{23} \\
\Phi_{31} & \Phi_{32} & \Phi_{33}
\end{bmatrix}
\]
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with
\[ \Phi_{11} = \text{diag}\left( \frac{2a_1\sigma_{11}}{(\sigma_{11} + a_1^2)^2}, \frac{2a_2\sigma_{22}}{(\sigma_{22} + a_2^2)^2}, \ldots, \frac{2a_p\sigma_{pp}}{(\sigma_{pp} + a_p^2)^2} \right), \]
\[ \Phi_{12} = \text{diag}\left( \frac{-a_1^2}{(\sigma_{11} + a_1^2)^2}, \frac{-a_2^2}{(\sigma_{22} + a_2^2)^2}, \ldots, \frac{-a_p^2}{(\sigma_{pp} + a_p^2)^2} \right), \]
\[ \Phi_{13} = \Phi_{23} = 0_{p \times 2}, \]
\[ \Phi_{21} = \text{diag}\left( \frac{a_1^2 - \sigma_{11}}{a_1^2}, \frac{a_2^2 - \sigma_{22}}{a_2^2}, \ldots, \frac{a_p^2 - \sigma_{pp}}{a_p^2} \right), \]
\[ \Phi_{22} = \text{diag}\left( \frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_p} \right), \]
\[ \Phi_{31} = \Phi_{32} = 0_{2 \times p}, \]
\[ \Phi_{33} = I_{2 \times 2}. \]

\[ \Psi_t = \text{var}(X_t | F_{t-1}), \quad \Psi_t = \text{var}(X_t^2 | F_{t-1}), \]

3.2 Weighted Conditional Least Squares Estimators

The Conditional Least Squares (CLS) estimates, in general, are not asymptotically efficient. Because \( \text{Var}(X_t | F_{t-1}, J_t) \), \( \text{Var}(X_t, X_t^2 | F_{t-1}, J_t) \), \( \text{Cov}(X_t, X_t^2 | F_{t-1}) \), \( \text{Var}(X_t^2 | F_{t-1}) \) and \( \text{Var}(X_t^2 | F_{t-1}, J_t) \) depend on \( X_{t-1}, X_{t-2}, \ldots, X_{t-p}, \phi_1, \phi_2, \ldots, \phi_p \), we can consider WCLS estimates to improve the efficiency. In this Section, we give these estimates.

Write
\[ \psi_t = E(u_t^2 | F_{t-1}), \quad \Psi_t = E(U_t^2 | F_{t-1}), \]
with \( U_t \) given in Subsection 3.1, then we have
\[ \psi_t = \text{var}(X_t | F_{t-1}), \]
\[ \Psi_t = \text{var}(X_t^2 | F_{t-1}) - 4E(X_t | F_{t-1})\text{cov}(X_t, X_t^2 | F_{t-1}) + 4(E(X_t | F_{t-1}))^2 \text{var}(X_t | F_{t-1}). \]

We can obtain the WCLS estimates by minimizing
\[ Q_3(\theta) = \sum_{t=p+1}^{n} \frac{u_t^2}{\psi_t} + \sum_{t=p+1}^{n} \frac{U_t^2}{\Psi_t} \]
with respect to \( \theta \). Since it is very difficult to derive explicit estimators of the parameters using an iterative method, we consider \( \theta \) replaced with the corresponding consistent estimates by other means. In particular, we may
choose to use the estimated versions of $\psi_t$ and $\Psi_t$ denoted by $\hat{\psi}_t$ and $\hat{\Psi}_t$ according to the CLS. Thus, we can derive $\hat{\theta}$. So with similar arguments as in Subsection 3.1, we can obtain the WCLS estimators of $a$, $\mu_{\varepsilon}$, $\sigma$ and $\sigma_{\varepsilon}^2$:

$$
\hat{a}_t^w = \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\psi_t} Y_t Y_t' - \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\psi_t} \right) \sum_{t=p+1}^{n} \frac{1}{\psi_t} Y_t \sum_{t=p+1}^{n} \frac{1}{\psi_t} Y_t' \right)^{-1} \times \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\psi_t} Y_t X_t - \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\psi_t} \right) \sum_{t=p+1}^{n} \frac{1}{\psi_t} Y_t \sum_{t=p+1}^{n} \frac{1}{\psi_t} X_t \right),
$$

(3.7)

$$
\hat{\mu}_{\varepsilon}^w = \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\psi_t} \right)^{-1} \sum_{t=p+1}^{n} \frac{1}{\psi_t} (X_t - Y_t' \hat{a}_t^w),
$$

(3.8)

$$
\hat{\sigma}^w = \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\Psi_t} Z_t Z_t' - \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\Psi_t} \right) \sum_{t=p+1}^{n} \frac{1}{\Psi_t} Z_t \sum_{t=p+1}^{n} \frac{1}{\Psi_t} Z_t' \right)^{-1} \times \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\Psi_t} Z_t \hat{H}_t^w - \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\Psi_t} \right) \sum_{t=p+1}^{n} \frac{1}{\Psi_t} Z_t \sum_{t=p+1}^{n} \frac{1}{\Psi_t} \hat{H}_t^w \right),
$$

(3.9)

$$
\hat{\sigma}_{\varepsilon}^2 = \left( \frac{1}{n} \sum_{t=p+1}^{n} \frac{1}{\Psi_t} \right)^{-1} \sum_{t=p+1}^{n} \frac{1}{\Psi_t} (\hat{H}_t^w - Z_t \hat{\sigma}^w),
$$

(3.10)

where $Y_t$ and $Z_t$ are given in Subsection 3.1, and

$$
\hat{H}_t^w = \left( X_t - \sum_{i=1}^{p} \hat{a}_i^w X_{t-i} - \hat{\mu}_{\varepsilon}^w \right)^2 + 2 \sum_{j=2}^{p} \sum_{i=1}^{j-1} \hat{a}_i^w \hat{a}_j^w X_{t-i} X_{t-j} - \sum_{i=1}^{p} X_{t-i} (\hat{a}_i^w - (\hat{a}_i^w)^2) .
$$

Thus, we can also obtain estimators $\hat{\theta}_w$ by using the similar equations given in equation (3.6). These estimators given in (3.7), (3.8), (3.9) and (3.10) are strongly consistent from the ergodic theorem. However, when the sample size $n$ is small, the mean squared errors (MSE) are large, a high proportion of estimates fall outside $\Omega$ in simulations and the above estimators cannot be guaranteed to be positive estimators for $\sigma$ and $\sigma_{\varepsilon}^2$, the results of which are presented in Section 4.
Remark 3.1. The proof of the strongly consistence of $\hat{\vartheta}^w$ is omitted here since it is difficult relatively.

Remark 3.2. The small sample size $n$ is usually less than 1000, which is presented in Section 4.

3.3 Maximum Quasi-likelihood Estimators

The MQEs for the DDRCINAR(p) model can be based on the p-dimensional stochastic process $\{X_t, X^2_t, \ldots, X^p_t\}$. The resulting system of estimating equations is given by:

$$
\sum_{t=p+1}^{n} B_1 \times B_2 \times B_3 = 0,
$$

where

$$
B_1 = \begin{bmatrix}
  e_{11} & e_{12} & \cdots & e_{1,2p+2} \\
  e_{21} & e_{22} & \cdots & e_{2,2p+2} \\
  \vdots & \vdots & \ddots & \vdots \\
  e_{p1} & e_{p2} & \cdots & e_{p,2p+2}
\end{bmatrix},
$$

$$
B_2 = \begin{bmatrix}
  v_{11} & v_{12} & \cdots & v_{1p} \\
  v_{21} & v_{22} & \cdots & v_{2p} \\
  \vdots & \vdots & \ddots & \vdots \\
  v_{p1} & v_{p2} & \cdots & v_{pp}
\end{bmatrix}^{-1},
$$

$$
B_3 = (X_t - E(X_t|\mathcal{F}_{t-1}), X^2_t - E(X^2_t|\mathcal{F}_{t-1}), \ldots, X^p_t - E(X^p_t|\mathcal{F}_{t-1}))',
$$

where

$$
e_{ij} = \frac{\partial E(X^i_t|\mathcal{F}_{t-1})}{\partial \alpha_j}, \quad e_{i,j+p} = \frac{\partial E(X^i_t|\mathcal{F}_{t-1})}{\partial \phi_j},$$

$$
e_{i,2p+1} = \frac{\partial E(X^i_t|\mathcal{F}_{t-1})}{\partial \mu_\varepsilon}, \quad e_{i,2p+2} = \frac{\partial E(X^i_t|\mathcal{F}_{t-1})}{\partial \sigma^2_\varepsilon},$$

$$
v_{ij} = \text{cov}(X^i_t, X^j_t|\mathcal{F}_{t-1}) = E(X^{i+j}_t|\mathcal{F}_{t-1}) - E(X^i_t|\mathcal{F}_{t-1})E(X^j_t|\mathcal{F}_{t-1}),$$

$i, j = 1, 2, \ldots, p$.

This nonlinear system of equations can be solved using an iterative method to obtain the MQEs $\hat{\vartheta}$ of parameter vector $\vartheta$ and the conditional moments used in equation (3.11) are given in the Appendix. Hutton, J. E. and Nelson, P. I. (1986) gave regularity conditions for the existence, strong consistency and asymptotic normality of the MQEs and showed that they are optimal in Godambe’s sense. However, for the DDRCINAR(p) model of order $p \geq 2$,
it is difficult to prove these regularity conditions because of the complexity of the algebraic expressions given in the appendix. Next, we consider some properties of MQEs when \( p = 1 \).

Let \( \xi = (\sigma_{11}, \eta, \sigma_{2}^2) '\), where \( \eta = a_1(1 - a_1) - \sigma_{11} \) and \( \beta = (a_1, \mu_\varepsilon)' \). Thus, the expression for one-step conditional variance

\[
V_\xi(X_t|X_{t-1}) = v_{11} = \sigma_{11}X_{t-1}^2 + \eta X_{t-1} + \sigma_{2}^2.
\]

According to (3.11) and (3.1), a set of MQEs estimating equations take the form:

\[
\begin{align*}
\sum_{t=2}^{n} V_\xi^{-1}(X_t|X_{t-1})(X_t - a_1 X_{t-1} - \mu_\varepsilon) &= 0, \\
\sum_{t=2}^{n} V_\xi^{-1}(X_t|X_{t-1})X_{t-1}(X_t - a_1 X_{t-1} - \mu_\varepsilon) &= 0.
\end{align*}
\]

(3.12)

Note that the presence of \( \xi \) in the expression for the conditional variance makes the corresponding estimating equations complicated and intractable in the general case. Therefore, we propose substituting a suitable consistent estimator \( \hat{\xi} \) of \( \xi \) obtained by other means and then solve the estimators of (3.12). This approach leads to the following closed form estimator of \( \beta \):

\[
\begin{pmatrix}
\hat{a}_1 \\
\hat{\mu}_\varepsilon
\end{pmatrix} = \left( \begin{array}{c}
\sum_{t=2}^{n} X_{t-1}V_\xi^{-1}(X_t|X_{t-1}) \\
\sum_{t=2}^{n} X_{t-1}^2 V_\xi^{-1}(X_t|X_{t-1})
\end{array} \right)^{-1} \left( \begin{array}{c}
\sum_{t=2}^{n} V_\xi^{-1}(X_t|X_{t-1})(X_t - a_1 X_{t-1} - \mu_\varepsilon) \\
\sum_{t=2}^{n} V_\xi^{-1}(X_t|X_{t-1})X_{t-1}(X_t - a_1 X_{t-1} - \mu_\varepsilon)
\end{array} \right)
\times \left( \begin{array}{c}
\sum_{t=2}^{n} X_tV_\xi^{-1}(X_t|X_{t-1}) \\
\sum_{t=2}^{n} X_tX_{t-1}V_\xi^{-1}(X_t|X_{t-1})
\end{array} \right).
\]

(3.13)

A consistent estimator of \( \xi \) is proposed next that can be used in (3.13).

**Proposition 3.1.** Let \( X_t \) be a DDRCINAR(1) model, then the following estimators are consistent:

\[
\hat{\sigma}_{\varepsilon}^2 = \frac{1}{n} \sum_{t=2}^{n} (X_t - \hat{a}_1 X_{t-1} - \hat{\mu}_\varepsilon)^2 - \frac{\hat{\sigma}_{11}}{n} \sum_{t=2}^{n} (X_{t-1}^2 - X_{t-1}) - \frac{\hat{a}_1 - \hat{a}_1^2}{n} \sum_{t=2}^{n} X_{t-1},
\]

\[
\hat{\eta} = \hat{a}_1 - \hat{a}_1^2 - \hat{\sigma}_{11},
\]

where \( \hat{a}_1, \hat{\sigma}_{11} \) and \( \hat{\mu}_\varepsilon \) are consistent estimators of \( a_1, \sigma_{11} \) and \( \mu_\varepsilon \). In practice, we can use the CLS or WCLS estimators of \( a_1, \sigma_{11} \) and \( \mu_\varepsilon \).
Proof. Let \( A_n = \frac{1}{n} \sum_{t=2}^{n} (X_t - a_1 X_{t-1} - \mu_\varepsilon)^2 \), \( B_n = \frac{1}{n} \sum_{t=2}^{n} X_{t-1}^2 \) and \( C_n = \frac{1}{n} \sum_{t=2}^{n} X_{t-1} \). By Theorem 1.1 of Billingsley, P. (1961), \( A_n \xrightarrow{a.s.} E((X_t - a_1 X_{t-1} - \mu_\varepsilon)^2) = \sigma_\varepsilon^2 + \sigma_{11}(\gamma_1 - \gamma_2) + (a_1 - a_1^2)\gamma_2 \), \( B_n \xrightarrow{a.s.} \gamma_1 \) and \( C_n \xrightarrow{a.s.} \gamma_2 \), where \( \gamma_1 = E(X_\infty^2), \gamma_2 = E(X_\infty) \) and \( X_\infty \) denotes the limiting random variable corresponding to the stationary of the process. Therefore,

\[
\hat{\sigma}_\varepsilon^2 = A_n - A_n + \frac{1}{n} \sum_{t=2}^{n} (X_t - \hat{a}_1 X_{t-1} - \hat{\mu}_\varepsilon)^2 - \frac{\hat{\sigma}_{11}}{n} \sum_{t=2}^{n} (X_{t-1}^2 - X_{t-1}) \\
- \frac{\hat{\alpha}_1 - \hat{a}_1}{n} \sum_{t=2}^{n} X_{t-1} \\
= A_n + (\hat{a}_1 - a_1)((\hat{a}_1 + a_1 - 2)B_n + 2\hat{\mu}_\varepsilon C_n) \\
+ (\hat{\mu}_\varepsilon - \mu_\varepsilon)(\hat{\mu}_\varepsilon + \mu_\varepsilon - 2(1 + a_1)C_n) - \hat{\sigma}_{11}(B_n - C_n) - (\hat{a}_1 - \hat{a}_1^2)C_n \\
\xrightarrow{p} \sigma_\varepsilon^2.
\]

Similar arguments lead to \( \hat{\eta} \xrightarrow{p} \eta \). \qed

Remark 3.3. \( \hat{a}_1 \) and \( \hat{\phi}_1 \) are also consistent by Proposition 3.1 and (3.6). Asymptotic normality of the MQEs estimators in (3.13) is established in the following theorem.

Theorem 3.2. The joint limit distribution of the MQEs estimators \( (\hat{a}_1, \hat{\mu}_\varepsilon) \) given by (3.13) is

\[
\sqrt{n} \left( \begin{array}{c} \hat{a}_1 - a_1 \\ \hat{\mu}_\varepsilon - \mu_\varepsilon \end{array} \right) \xrightarrow{d} N(0, T^{-1}(\xi)Q(\xi)T^{-1}(\xi)), \quad n \to +\infty,
\]

where

\[
Q(\xi) = \begin{pmatrix} T_1(\xi) & T_3(\xi) \\ T_3(\xi) & T_2(\xi) \end{pmatrix},
\]

\[
T^{-1}(\xi) = (T_3^2(\xi) - T_1(\xi)T_2(\xi))^{-1} \begin{pmatrix} T_3(\xi) & -T_1(\xi) \\ -T_2(\xi) & T_3(\xi) \end{pmatrix},
\]

where \( T_1(\xi) = E[V_\varepsilon^{-1}(X_2|X_1)], T_2(\xi) = E[X_2^2V_\varepsilon^{-1}(X_2|X_1)] \) and \( T_3(\xi) = E[X_1V_\varepsilon^{-1}(X_2|X_1)] \).
Proof. First, we suppose $\xi$ is known. For the following estimation equations:

$$S_n^{(1)}(\xi, \beta) = \sum_{t=2}^{n} V^{-1}_\xi(X_t|X_{t-1})(X_t - a_1 X_{t-1} - \mu_\varepsilon),$$

$$S_n^{(2)}(\xi, \beta) = \sum_{t=2}^{n} V^{-1}_\xi(X_t|X_{t-1})X_{t-1}(X_t - a_1 X_{t-1} - \mu_\varepsilon),$$

we have

$$E[V^{-1}_\xi(X_t|X_{t-1})(X_t - a_1 X_{t-1} - \mu_\varepsilon)|F_{t-1}] = V^{-1}_\xi(X_t|X_{t-1})E[(X_t - a_1 X_{t-1} - \mu_\varepsilon)|F_{t-1}] = 0$$

and

$$E[S_n^{(1)}(\xi, \beta)|F_{t-1}] = S_{t-1}^{(1)}(\xi, \beta).$$

Thus, $\{S_n^{(1)}(\xi, \beta), F_t, t \geq 0\}$ is a martingale. By Theorem 1.1 of Billingsley, P. (1961),

$$\frac{1}{n} \sum_{t=2}^{n} V^{-2}_\xi(X_t|X_{t-1})(X_t - a_1 X_{t-1} - \mu_\varepsilon)^2 \rightarrow \text{a.s.} E[V^{-2}_\xi(X_2|X_1)(X_2 - a_1 X_1 - \mu_\varepsilon)^2]$$

$$= E[E(V^{-2}_\xi(X_2|X_1)(X_2 - a_1 X_1 - \mu_\varepsilon)^2|X_1)] = E[V^{-1}_\xi(X_2|X_1)] = T_1(\xi).$$

By Corollary 3.2 of Hall, P. and Heyde, C. C. (1980), the martingale MQEs applies and we get

$$\frac{1}{\sqrt{n}} S_n^{(1)}(\xi, \beta) \overset{d}{\rightarrow} N(0, T_1(\xi)), \quad n \rightarrow +\infty.$$ 

Similarly,

$$\frac{1}{n} \sum_{t=2}^{n} V^{-2}_\xi(X_t|X_{t-1})X_{t-1}^2(X_t - a_1 X_{t-1} - \mu_\varepsilon)^2 \rightarrow \text{a.s.} E(V^{-2}_\xi(X_2|X_1)X_1^2(X_2 - a_1 X_1 - \mu_\varepsilon)^2)$$

$$= E[E(V^{-2}_\xi(X_2|X_1)X_1^2(X_2 - a_1 X_1 - \mu_\varepsilon)^2|X_1)] = E[X_1^2 V^{-1}_\xi(X_2|X_1)] = T_2(\xi)$$

and

$$\frac{1}{\sqrt{n}} S_n^{(2)}(\xi, \beta) \overset{d}{\rightarrow} N(0, T_2(\xi)), \quad n \rightarrow +\infty.$$
Again by Cramer-Wold device, for any $c = (c_1, c_2)'$, $c_1$ and $c_2 \in \mathcal{R}$ are not both 0. When $n \to +\infty$, we have

$$
\frac{c'}{\sqrt{n}} \left( \begin{array}{c}
S_n^{(1)}(\xi, \beta) \\
S_n^{(2)}(\xi, \beta)
\end{array} \right) \overset{d}{\to} N(0, E[V_\xi^{-2}(X_2|X_1)(c_2X_1 + c_1)^2(X_2 - a_1X_1 - \mu_\epsilon)^2]),
$$

implying

$$
\frac{1}{\sqrt{n}} \left( \begin{array}{c}
S_n^{(1)}(\hat{\xi}, \beta) \\
S_n^{(2)}(\hat{\xi}, \beta)
\end{array} \right) \overset{d}{\to} N \left( \left( \begin{array}{c}0 \\
T_1(\xi) T_2(\xi)
\end{array} \right), \left( \begin{array}{cc}
T_3(\xi) & T_3(\xi) \\
T_2(\xi) & T_2(\xi)
\end{array} \right) \right), \quad n \to +\infty. \; (3.14)
$$

where $T_3(\xi) = E[V_\xi^{-2}(X_2|X_1)X_1(X_2 - a_1X_1 - \mu_\epsilon)^2] = E[X_1V_\xi^{-1}(X_2|X_1)]$.

Now, we replace $V_\xi^{-2}(X_t|X_{t-1})$ by $V_\xi^{-2}(X_t|X_{t-1})$, where $\hat{\xi}$ is a consistent estimator of $\xi$. Then we want

$$
\frac{1}{\sqrt{n}} \left( \begin{array}{c}
S_n^{(1)}(\hat{\xi}, \beta) \\
S_n^{(2)}(\hat{\xi}, \beta)
\end{array} \right) \overset{d}{\to} N \left( \left( \begin{array}{c}0 \\
T_1(\xi) T_2(\xi)
\end{array} \right), \left( \begin{array}{cc}
T_3(\xi) & T_3(\xi) \\
T_2(\xi) & T_2(\xi)
\end{array} \right) \right), \quad n \to +\infty. \; (3.15)
$$

To obtain this we need to prove that

$$
\frac{1}{\sqrt{n}} S_n^{(i)}(\hat{\xi}, \beta) \overset{p}{\to} 0, \quad i = 1, 2. \; (3.16)
$$

Let $R_n(\xi) = (1/\sqrt{n})S_n^{(1)}(\xi, \beta)$. Then $\forall \epsilon > 0$ and $\delta > 0$ such that $\xi - \delta 1 > 0$, where $1$ is the unit vector, we have

$$
P(\{R_n(\hat{\xi}) - R_n(\xi) > \epsilon\} \leq P(\{\hat{\sigma}_{11} - \sigma_{11} > \delta\} + p(|\hat{\eta} - \eta| > \delta)
+ p(|\hat{\sigma}_2^2 - \sigma_2^2| > \delta)
+ P(\sup_{\{\sigma_1 - \sigma_{11} < \delta, \eta_1 - \eta < \delta, \sigma_2^2 - \sigma_2^2 < \delta\}} |R_n(\xi_1) - R_n(\xi)| > \epsilon),
$$

where $\xi_1 = (\sigma_1, \eta_1, \sigma_2^2)'$. Let $D = \{\{\sigma_1 - \sigma_{11} < \delta, |\eta_1 - \eta| < \delta, |\sigma_2^2 - \sigma_2^2| < \delta\}$. If $\hat{\xi}$ is a consistent estimator of $\xi$, then we just need to prove that

$$
P\left(\sup_{D} |R_n(\xi_1) - R_n(\xi)| > \epsilon\right) \overset{p}{\to} 0.
By Markov inequality,

\[ P \left( \sup_{D} \left| R_n(\xi_1) - R_n(\xi) \right| > \epsilon \right) \leq \frac{1}{\epsilon^2} E \left( \sup_{D} (R_n(\xi_1) - R_n(\xi))^2 \right) \]

\[ = \frac{1}{\epsilon^2} E \left( \sup_{D} \frac{1}{n} \sum_{t=2}^{n} (V_{t-1}^{-1}(X_t|X_{t-1}) - V_{t-1}^{-1}(X_{t-1}|X_{t-1}))^2 (X_t - a_1X_{t-1} - \mu_\xi)^2 \right) \]

\[ = \frac{1}{\epsilon^2} E \left( \sup_{D} (V_{t-1}^{-1}(X_2|X_1) - V_{t-1}^{-1}(X_2|X_1))^2 (X_2 - a_1X_1 - \mu_\xi)^2 \right) \]

\[ = \frac{1}{\epsilon^2} E \left( \sup_{D} \frac{(\sigma_1 - \sigma_{11})X_1^2 + (\eta_1 - \eta)X_1 + (\sigma_2^2 - \sigma_\xi^2))^2}{V_{t-1}^2(X_2|X_1)V_{t-1}^2(X_2|X_1)} (X_2 - a_1X_1 - \mu_\xi)^2 \right) \]

\[ \leq \frac{1}{\epsilon^2} \sup_{D} \{(\sigma_1 - \sigma_{11})^2c_1 + (\eta_1 - \eta)^2c_2 + (\sigma_2^2 - \sigma_\xi^2)^2c_3 + 2c_4|\sigma_1 - \sigma_{11}|(\eta_1 - \eta)| + 2c_5|\sigma_1 - \sigma_{11}|(\sigma_2^2 - \sigma_\xi^2)| + 2c_6|\eta_1 - \eta|(\sigma_2^2 - \sigma_\xi^2)| = C\delta^2, \]

where \(\{c_i, i = 1, \cdots, 6\}\) are finite moments and \(C\) is a positive constant. Similar argument can be used for \(\tilde{S}_n^{(2)}(\xi, \beta)\). When \(\delta\) goes to zero, we get our assertion which in turn establishes (3.15). Similarly, we have

\[ \frac{1}{n} \sum_{t=2}^{n} V_{t-1}^{-1}(X_t|X_{t-1}) - \frac{1}{n} \sum_{t=2}^{n} V_{t-1}^{-1}(X_t|X_{t-1}) \xrightarrow{p} 0, \]

\[ \frac{1}{n} \sum_{t=2}^{n} X_{t-1}V_{t-1}^{-1}(X_t|X_{t-1}) - \frac{1}{n} \sum_{t=2}^{n} X_{t-1}V_{t-1}^{-1}(X_t|X_{t-1}) \xrightarrow{p} 0, \]

\[ \frac{1}{n} \sum_{t=2}^{n} X_{t-1}V_{t-1}^{-1}(X_t|X_{t-1}) - \frac{1}{n} \sum_{t=2}^{n} X_{t-1}V_{t-1}^{-1}(X_t|X_{t-1}) \xrightarrow{p} 0. \]

Therefore, by the above and Theorem 1.1 of Billingsley, P. (1961), we have

\[ (A_1 - A_2)\frac{1}{n} \sum_{t=2}^{n} X_{t-1}V_{t-1}^{-1}(X_t|X_{t-1}) \xrightarrow{p} (T_3^2(\xi) - T_4(\xi)T_2(\xi))^{-1} \left( \begin{array}{cc} T_3(\xi) & -T_1(\xi) \\ -T_2(\xi) & T_3(\xi) \end{array} \right) = T^{-1}(\xi), \]

where

\[ A_1 = \left( \frac{1}{n} \sum_{t=2}^{n} X_{t-1}V_{t-1}^{-1}(X_t|X_{t-1}) \right)^2, \]
Estimation of parameters in the DDRCINAR(p) model

\[ A_2 = \left( \frac{1}{n} \sum_{t=2}^{n} V^{-1}_{\tilde{\xi}}(X_t|X_{t-1}) \right) \left( \frac{1}{n} \sum_{t=2}^{n} X_{t-1}^2 V^{-1}_{\tilde{\xi}}(X_t|X_{t-1}) \right) \]

\[ A_3 = \begin{pmatrix}
\frac{1}{n} \sum_{t=2}^{n} X_{t-1} V^{-1}_{\tilde{\xi}}(X_t|X_{t-1}) & -\frac{1}{n} \sum_{t=2}^{n} V^{-1}_{\tilde{\xi}}(X_t|X_{t-1}) \\
-\frac{1}{n} \sum_{t=2}^{n} X_{t-1}^2 V^{-1}_{\tilde{\xi}}(X_t|X_{t-1}) & \frac{1}{n} \sum_{t=2}^{n} X_{t-1} V^{-1}_{\tilde{\xi}}(X_t|X_{t-1})
\end{pmatrix}. \]

After some algebra, we have

\[ \left( \tilde{a}_1 - a_1 \quad \tilde{\mu}_\varepsilon - \mu_\varepsilon \right) = n^{-1} (A_1 - A_2)^{-1} \times A_3 \times \begin{pmatrix} S_n^{(1)}(\tilde{\xi},\beta) \\ S_n^{(2)}(\tilde{\xi},\beta) \end{pmatrix} \]

Therefore, by (3.15) and (3.17),

\[ \sqrt{n} \left( \begin{pmatrix} \tilde{a}_1 - a_1 \\ \tilde{\mu}_\varepsilon - \mu_\varepsilon \end{pmatrix} \right) \overset{d}{\rightarrow} N(0, T^{-1}(\xi)'Q(\xi)T^{-1}(\xi)), \quad n \rightarrow +\infty, \]

where

\[ Q(\xi) = \begin{pmatrix} T_1(\xi) & T_3(\xi) \\ T_3(\xi) & T_2(\xi) \end{pmatrix}. \]

\[ \square \]

**Remark 3.4.** When \( p = 1 \), the MQEs is equivalent to modified quasi-likelihood (MQL) estimation method proposed by Zheng, H., Basawa, I. V. and Datta, S. (2007). Thus, the proof of the above theorem can be also obtained by Zheng, H., Basawa, I. V. and Datta, S. (2007).

### 4 Simulations

Consider the model 2.1 with \( p = 2 \), that is, DDRCINAR(2) model, where \( \varepsilon_t \) is an i.i.d. poisson sequence with mean \( \lambda \), and

\[ E(A'_t \otimes A_t) = \begin{pmatrix}
\alpha_1 \phi_1^2 & 0 & \alpha_1 \phi_1 & \alpha_2 \phi_2 \\
\alpha_1 \phi_1 & 0 & 1 & 0 \\
0 & \alpha_2 \phi_2 & 0 & 0 \\
\alpha_2 \phi_2 & 0 & 0 & 0
\end{pmatrix}. \]

By Theorem 2.1, the conditions for the existence of this stationary and ergodic DDRCINAR(2) model are: \( 0 \leq \alpha_1, \alpha_2 \leq 1; 0 < \alpha_1 + \alpha_2 < 1; 0 \leq \phi_1, \phi_2 \leq 1; \alpha_1 \phi_1 + \alpha_2 \phi_2 < 1 \) and maximum absolute eigenvalue of \( E(A'_t \otimes A_t) < 1 \). These conditions define the parameter space \( \Omega \) used here.
A simulation study was conducted by generating DDRCINAR(2) processes, each of which are from four samples of parameter values for $\alpha_i$, $\phi_i$ and $\lambda$, namely

(a) sample 1: $\alpha_1 = 0.4, \alpha_2 = 0.5, \phi_1 = 0.6, \phi_2 = 0.7$ and $\lambda = 1$,
(b) sample 2: $\alpha_1 = 0.4, \alpha_2 = 0.5, \phi_1 = 0.6, \phi_2 = 0.7$ and $\lambda = 2$,
(c) sample 3: $\alpha_1 = 0.25, \alpha_2 = 0.25, \phi_1 = 0.5, \phi_2 = 0.5$ and $\lambda = 1$,
(d) sample 4: $\alpha_1 = 0.25, \alpha_2 = 0.25, \phi_1 = 0.5, \phi_2 = 0.5$ and $\lambda = 2$.

Figure 1 on page 20 gives four typical sample paths for a sample size 200 about the DDRCINAR(2) model. We use R for the random number generation and sample sizes $n = 200, 500, 1000, 10,000, 30,000$ and 500 replications were used. For WCLS and MQE estimates, $\psi_t, \Psi_t$ and $v_{ij}$ are estimated by using CLS. We use the mean squared errors (MSE), i.e.
Estimation of parameters in the DDRCINAR(p) model

Table 1  Mean values of three sets of estimates for samples 1 and 2.

<table>
<thead>
<tr>
<th></th>
<th>$n$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\lambda$</th>
</tr>
</thead>
<tbody>
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<td>0.5001</td>
<td>0.5985</td>
<td>0.7004</td>
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<tr>
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<td>0.5009</td>
<td>0.6022</td>
<td>0.6995</td>
<td>0.9988</td>
</tr>
<tr>
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<tr>
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</tr>
<tr>
<td>MQE</td>
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<td>0.4997</td>
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</tr>
<tr>
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<td>0.5003</td>
<td>0.6036</td>
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</tr>
</tbody>
</table>

$\alpha_1 = 0.4, \alpha_2 = 0.5, \phi_1 = 0.6, \phi_2 = 0.7, \lambda = 1$

$\alpha_1 = 0.4, \alpha_2 = 0.5, \phi_1 = 0.6, \phi_2 = 0.7, \lambda = 2$

$\frac{1}{n} \sum_{j=1}^{n} (\varphi_{est} - \varphi)^2$, \hspace{1cm} (4.1)

to evaluate the performance of the estimators, where $n$ is the number of realizations and $\varphi_{est}$ denotes any estimator of $\varphi$. Table 1 on page 21 enumerates the estimates of parameters for samples 1 and 2, with similar results given for samples 3 and 4 in Table 3 on page 23. The representative results about MSE and the per cent lying in $\Omega$ for samples 1, 2, 3 and 4 are summarized in Tables 2 on page 22 and 4 on page 24, respectively.

Values outside the allowed range for $\varphi$ might easily be obtained for small sample sizes of $n = 200, 500$ and 1000. By (3.6), the estimates of $\alpha_i$ and $\phi_i$ are decided by $a$ and $\sigma$. Therefore, when the sample size is very small, there emerge negative estimated values of $\alpha_i$ and $\phi_i$. We thus need to adjust negative estimates in a somewhat ad hoc manner. In Tables 3 on page 23 and 4 on page 24, such estimated have been adjusted by taking account that the restrictions on $\alpha_i$ and $\phi_i$ imply,
Table 2  MSE and per cent within parenthese space of three sets of estimates for samples 1 and 2.

<table>
<thead>
<tr>
<th>n</th>
<th>α₁</th>
<th>α₂</th>
<th>φ₁</th>
<th>φ₂</th>
<th>λ</th>
<th>% in Ω</th>
</tr>
</thead>
<tbody>
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<td>10,000</td>
<td>0.0005</td>
<td>0.0004</td>
<td>0.0007</td>
<td>0.0004</td>
<td>0.0008</td>
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</tr>
<tr>
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<td>0.0001</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0003</td>
<td>100.00</td>
</tr>
<tr>
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<td>0.0005</td>
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<td>0.0001</td>
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<td>100.00</td>
</tr>
<tr>
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<td>0.0001</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
<td>100.00</td>
</tr>
</tbody>
</table>

\( \alpha_1 = 0.4, \alpha_2 = 0.5, \phi_1 = 0.6, \phi_2 = 0.7, \lambda = 1 \)

CLS

<table>
<thead>
<tr>
<th>n</th>
<th>α₁</th>
<th>α₂</th>
<th>φ₁</th>
<th>φ₂</th>
<th>λ</th>
<th>% in Ω</th>
</tr>
</thead>
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<td>0.0003</td>
<td>0.0002</td>
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<td>0.0002</td>
<td>0.0002</td>
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<td>0.0008</td>
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<td>0.0000</td>
<td>0.0003</td>
<td>100.00</td>
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</tbody>
</table>

\( \alpha_1 = 0.4, \alpha_2 = 0.5, \phi_1 = 0.6, \phi_2 = 0.7, \lambda = 2 \)

WCLS

\[
\begin{aligned}
0 < a_1 + a_2 < 1, \\
\sigma_{11}\sigma_{22} > a_1^2 a_2^2, \\
0 < \sigma_{ii} < 0.25, \\
\sigma_{ii} < a_i(1 - a_i), \ i = 1, 2.
\end{aligned}
\]

Thus if \( \hat{a}_1 + \hat{a}_2 > 1 \), \( \hat{a}_i \) have been replaced by \( \hat{a}_i/(\hat{a}_1 + \hat{a}_2) \), and other constraints and parameters can be made with similar adjustments.

As the sample size increases, three sets of estimates seem to converge to the true parameter values, indicating consistency. However, the MQEs seem to fall within \( \Omega \) with a higher proportion of times than the existing estimators of CLS and WCLS, which indicate an improvement. Observing Tables 2 on page 22 and 4 on page 24, it is conclusion that MQEs dominate CLS and WCLS in terms of the MSE, which is accordance with our expectation. Meanwhile, when the sample size is increased, it is better to estimate the parameters for these three estimation methods, which shows large sample size may be needed to obtain reasonable results.

In order to more fully compare the three sets of estimators in terms of the percentage of estimates that fall within \( \Omega \), we calculate and examine the \( \Omega \) of the two estimators for a range of different parameter values when
Table 3  Mean values of three sets of estimates for samples 3 and 4.

<table>
<thead>
<tr>
<th>n</th>
<th>α₁</th>
<th>α₂</th>
<th>φ₁</th>
<th>φ₂</th>
<th>λ</th>
</tr>
</thead>
<tbody>
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<td>0.4280</td>
<td>0.3991</td>
<td>0.4010</td>
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<tr>
<td>500</td>
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<td>0.4292</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>0.4998</td>
<td>0.9999</td>
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</tbody>
</table>

CLS

<table>
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<tr>
<th>n</th>
<th>α₁</th>
<th>α₂</th>
<th>φ₁</th>
<th>φ₂</th>
<th>λ</th>
</tr>
</thead>
<tbody>
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<tr>
<td>500</td>
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<td>0.3839</td>
<td>0.4268</td>
<td>0.4293</td>
<td>0.9659</td>
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<tr>
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<td>0.3249</td>
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<td>0.4582</td>
<td>0.9804</td>
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<tr>
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<td>0.2569</td>
<td>0.4973</td>
<td>0.4947</td>
<td>1.0004</td>
</tr>
<tr>
<td>30,000</td>
<td>0.2507</td>
<td>0.2507</td>
<td>0.5005</td>
<td>0.5014</td>
<td>0.9999</td>
</tr>
</tbody>
</table>

WCLS

<table>
<thead>
<tr>
<th>n</th>
<th>α₁</th>
<th>α₂</th>
<th>φ₁</th>
<th>φ₂</th>
<th>λ</th>
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</thead>
<tbody>
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<td>0.9675</td>
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<tr>
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</tr>
<tr>
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<td>0.2521</td>
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</table>

MQE

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<th>α₂</th>
<th>φ₁</th>
<th>φ₂</th>
<th>λ</th>
</tr>
</thead>
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<tr>
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CLS

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<th>α₂</th>
<th>φ₁</th>
<th>φ₂</th>
<th>λ</th>
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WCLS

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<th>α₂</th>
<th>φ₁</th>
<th>φ₂</th>
<th>λ</th>
</tr>
</thead>
<tbody>
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<tr>
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</table>
Table 4 \textit{MSE and per cent within parentese space of three sets of estimates for samples 3 and 4.}

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\lambda$</th>
<th>$%$ in $\Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1 = \alpha_2 = 0.25, \phi_1 = \phi_2 = 0.5, \lambda = 1$</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>CLS</td>
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<td></td>
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</tr>
<tr>
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<td>0.0011</td>
<td>0.0001</td>
<td>100.00</td>
</tr>
<tr>
<td>MQE</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>99.0</td>
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</table>
\( \phi_1 = \phi_2 = 0.6, \lambda = 1, \) and \( \phi_1 = \phi_2 = 0.6, \lambda = 2. \) The sum of \( \alpha_1 \) and \( \alpha_2 \) is confined within the range of \([0.0,1.0]\). And, for each of these two parameters, different values range from 0.0 to 1.0, on a grid of 0.10. All possible samples of \( \alpha_1 \) and \( \alpha_2 \) are examined.

Tables 5 on page 25 and 6 on page 26 show the proportion of estimates for the parameters, \( \hat{\alpha}_1, \hat{\alpha}_2, \hat{\phi}_1, \hat{\phi}_2 \) and \( \hat{\lambda} \). Obviously, when \( \alpha_1 \) or \( \alpha_2 \) is 0 or the sum of \( \alpha_1 \) and \( \alpha_2 \) is 1, the percentage of in-range estimates is very small, which indicates that the problem of out-of-range estimates is severe near the boundaries of \( \Omega \). Generally, it is true that the three estimation methods improve when \( \alpha_1 \) or \( \alpha_2 \) or both increase(s). Specifically, it can be seen, from Tables 5 on page 25 and 6 on page 26, that MQEs perform better than the estimators of CLS and WCLS in terms of the proportion of within-\( \Omega \) estimates.

We also simulated other representative parameter samples. We find that good estimates and high proportion of within-\( \Omega \) estimates can be derived when \( \alpha_1 \) and \( \alpha_2 \) are both large, and poor estimates and low proportion of within-\( \Omega \) estimates are derived when \( \alpha_1 \) or \( \alpha_2 \) is small.

5 Real Data Analysis

In this Section, we will show how the model and methods from Section 3 can be applied to two real data time series. Moreover, we will consider two kinds of criteria to compare different models to the real data sets. The first kind is the root mean squared (RMS) errors defined by

\[
\text{RMS} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{X}_i)^2},
\]

and the second is defined as

\[
\text{MSE} = \frac{1}{m} \sum_{i=1}^{m} (X_{n-m+i} - \hat{X}_{n-m+i})^2,
\]

where \( \hat{X}_k = E(X_k|\mathcal{F}_{t-1}) \) and take \( m = 30 \). The MSE criteria is studied by Li, Q., Lian, H. and Zhu, F. (2016). The predictive performance of models is evaluated according to the two criteria.

5.1 Epileptic seizure counts analysis

Figure 2 Seizure counts plot.

Figure 3 Seizure counts ACF and PACF plots.

Table 7 Parameter estimation for model 7.1

<table>
<thead>
<tr>
<th>Method</th>
<th>Parameter</th>
<th>( \alpha_6 )</th>
<th>( \alpha_{14} )</th>
<th>( \phi_6 )</th>
<th>( \phi_{14} )</th>
<th>( \mu_\varepsilon )</th>
</tr>
</thead>
<tbody>
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<td>CLS</td>
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<td>0.3664</td>
<td>1.0423</td>
<td>0.7001</td>
<td>0.2342</td>
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<td>WCLS</td>
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<td>0.2397</td>
<td>1.3373</td>
<td>0.8756</td>
<td>0.1843</td>
<td>0.4412</td>
</tr>
<tr>
<td>MQE</td>
<td></td>
<td>0.2468</td>
<td>0.8390</td>
<td>0.8946</td>
<td>0.2962</td>
<td>0.4421</td>
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</table>

Using the methods proposed in Section 3, the following results are given in Table 7 on page 29.

Since \( \hat{\alpha}_{14} > 1 \) for CLS and WCLS methods, and \( \hat{\alpha}_6 + \hat{\alpha}_{14} > 1 \) for MQE in Table 7 on page 29, we should treat \( \phi_{t,14} \) as the fixed coefficient, i.e. take \( \sigma_{14} = 0 \). Then, using WCLS method, we obtain the variance estimation of \( \sigma_\varepsilon^2 \) is that \( \hat{\sigma}_\varepsilon^2 = 0.5015 \). The difference between \( \hat{\sigma}_\varepsilon^2 \) and \( \hat{\mu}_\varepsilon \) is 0.0603, which indicates it is reasonable that we assume the error \( \varepsilon_t \) has Poisson distribution with expectation 0.4421. Thus, our model for these data is degenerated to

\[
X_t = \phi_{t,6} \circ X_{t-6} + \phi_{t,14} \circ X_{t-14} + \varepsilon_t, \quad (5.5)
\]
Figure 4 Diagnostic checking plots under different models for the monthly Seizure data. (a),(e) Standardized residuals; (b),(f) Histograms of standardized residuals; (c),(g) ACF plots of residuals; (d),(h) PACF plots of residuals.

where the distribution of \( \{ \phi_t \} \) is given by
\[
\begin{align*}
    p(\phi_t = \phi_6) &= \alpha_6; \\
    p(\phi_t = 0) &= 1 - \alpha_6.
\end{align*}
\]  

(5.6)

According to the above assumptions, maximum quasi-likelihood estimators for Model (5.5) are given by \( \hat{\alpha}_6 = 0.2459, \hat{\phi}_6 = 0.8942, \hat{\phi}_{14} = 0.2455 \) and \( \hat{\mu}_\varepsilon = 0.4446 \). Then by (5.1) and (5.2), we have
\[
\begin{align*}
    \text{RMS} &= 0.8837, \\
    \text{MSE} &= 0.5878.
\end{align*}
\]

If we use the fixed coefficient model, where \( \varepsilon_t \) is poisson-distributed, then we have the following results for the MQE method:
\[
\hat{\phi}_6 = 0.2277, \quad \hat{\phi}_{14} = 0.2271, \quad \hat{\mu}_\varepsilon = 0.4614,
\]

similarly, we have
\[
\begin{align*}
    \text{RMS} &= 0.8839, \\
    \text{MSE} &= 0.5919.
\end{align*}
\]
The fitting results are summarized in Figure 4 on page 30. Figure 4 on page 30 shows the standardized residuals, the histograms of standardized residuals, ACF and PACF plots of residuals under two models. As is known in Figure 4 on page 30, the residuals are stationary series. Furthermore, the residual mean and variance of Model (5.5) and fixed-coefficient model are (-0.0119, 0.7882) and (-0.0228, 0.7882), respectively, which show that the Model (5.5) is closest to a normal distribution relatively.

From the above results, we can see that the random coefficient model has the smallest RMS and MSE. On the one hand, based on the RMS and MSE alone, one may prefer to select the coefficient model for these data. On the other hand, one may prefer to select the coefficient model for these data because of the innovation that the autoregressive parameters are dependence-driven random variables with a joint distribution function.

Remark 5.1. The confidence internals in Figure 3 on page 28 are

\((-2/\sqrt{n}, 2/\sqrt{n}) = (-0.1818, 0.1818),\)

which can be obtained by Cryer, J. D. and Chan, K. S. (2008). The confidence internals in the following ACF and PACF plots can be derived by the same way.

5.2 Precinct rape counts analysis

The data are obtained from the rape data section of the Forecasting Principles site (http://www.forecastingprinciples.com). There are 132 observations, starting in January 1991 and ending in December 2001. Note that the counts vary from 0 to 9. The sample mean and variance are 2.2348 and 2.9902, respectively. The plots of the time series, its ACF and PACF are given in Figure 5 on page 32. Analyzing the diagrams we conclude that the first-order autoregressive model is appropriate for the given data series. Therefore, we consider two models for the data. They are:

Model I.

\[ X_t = \phi_{t,10} \circ X_{t-10} + \varepsilon_t, \] (5.7)

the distribution of \( \{\phi_{t,10}\} \) is given by

\[
\begin{align*}
p(\phi_{t,10} = \phi_{t10}) &= \alpha_{t10}; \\
p(\phi_{t,10} = 0) &= 1 - \alpha_{t10}.
\end{align*}
\]

Where \( \{\varepsilon_t\} \) is an i.i.d poisson sequence with mean \( \lambda \) and \( \phi_{t10} \) is in [0, 1).

Model II.

\[ X_t = \varphi_{t10} \circ X_{t-10} + \varepsilon_t \]
where \( \{\varepsilon_t\} \) is an i.i.d poisson sequence with mean \( \mu \) and \( \varphi_{10} \) is fixed in \([0, 1)\).

The fitting results are summarized in Figure 6 on page 32 and Table 8 on page 33. Figure 6 on page 32 shows the standardized residuals, the histograms of standardized residuals, ACF and PACF plots of residuals under two models. As is known in Figure 6 on page 32, the residuals are stationary.
Table 8: Parameters, RMS and MSE

<table>
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<tr>
<th></th>
<th>MQE</th>
<th>CLS</th>
<th>WCLS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model I</td>
<td>\hat{\alpha}_{01}^{MQE} = 0.3709</td>
<td>\hat{\alpha}_{01}^{CLS} = 0.5753</td>
<td>\hat{\alpha}_{01}^{WCLS} = 0.4079</td>
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<tr>
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<td>\hat{\phi}_{01}^{MQE} = 0.5198</td>
<td>\hat{\phi}_{01}^{CLS} = 0.4408</td>
<td>\hat{\phi}_{01}^{WCLS} = 0.5425</td>
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<tr>
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<td>\hat{\lambda}_{01}^{MQE} = 1.7614</td>
<td>\hat{\lambda}_{01}^{CLS} = 1.6231</td>
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<tr>
<td>RMS</td>
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<tr>
<td>MSE</td>
<td>1.7195</td>
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<tr>
<td>Model II</td>
<td>\hat{\phi}_{01}^{MQE} = 0.1645</td>
<td>\hat{\phi}_{01}^{CLS} = 0.2536</td>
<td>\hat{\phi}_{01}^{WCLS} = 0.2341</td>
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<tr>
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<td>\hat{\mu}_{01}^{MQE} = 1.8287</td>
<td>\hat{\mu}_{01}^{CLS} = 1.6231</td>
<td>\hat{\mu}_{01}^{WCLS} = 1.6672</td>
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<td>RMS</td>
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<tr>
<td>MSE</td>
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</table>

series. From the histograms of figure 6 on page 32, the two models are well closed to a normal distribution. In Table 8 on page 33, we also give the predicted values RMS and MSE for each model. Moreover, The intuitionistic reason that the CLS estimators of \( \lambda \) and \( \mu \) are relatively small is as follows:

It is easy to obtain the CLS estimations,

\[
\hat{\alpha}_{CLS} = \left( \frac{1}{n - 10} \sum_{t=11}^{n} Y_t Y'_t - \frac{1}{n - 10} \sum_{t=11}^{n} Y_t \sum_{t=11}^{n} Y'_t \right)^{-1} \\
\times \left( \sum_{t=11}^{n} Y_t X_t - \frac{1}{n - 10} \sum_{t=11}^{n} Y_t \sum_{t=11}^{n} X_t \right),
\]

where \( n = 132 \), the data which we use to estimate \( \lambda \) are \( X_{11}, X_{12}, \ldots, X_{132} \) without \( X_1, X_2, \ldots, X_{10} \), thus some important "information" are lost. Then a similar argument can be applied to WCLS. But MQEs are based on the 10-dimensional stochastic process \( \{X_t, X_2^t, \ldots, X_{10}^t\} \), where the value of \( t \) is from 11 to 132. Thus, MQEs may derive more "information". Therefore, we recommend MQE method to analyse the real data. As can be seen, RMS and MSE are smaller for the Model I than Model II. For this data, all criteria show that Model I performs best.

6 Summary and conclusion

In this paper, we have introduced a \( p \)-th order dependence-driven random coefficient integer-valued autoregressive model for count data. The autoregressive coefficient is allowed to vary randomly over time. The stationarity
and ergodicity of the process are established. MQE, CLS and WCLS methods are used to estimate the parameters. Some of their asymptotic properties are obtained.

In the simulation study, we have shown three estimation methods for the DDRCINAR(p) model. And we conclude that a very large sample size may be needed to obtain reasonable results. Without considering the time factor, we recommend to use MQE to estimate the parameters in the DDRCINAR(p) model. We also consider the proportion of in-range parameter estimates of the DDRCINAR(p) model. It is concluded that the MQE method performs better than CLS and WCLS methods on the part of the proportion of within-Ω estimates, especially near the boundaries of the parameter space. The model is applied to two real data sets. It is shown that the dependence-driven random coefficient model (DDRCINAR(p)) is suitable for the real data sets.

Appendix A: Appendix section

In Section 2, we introduce our model DDRCINAR(p). Next, we give the conditional moments used in Section 4. From equation (2.1) we have

\[
X_t^m = \left( \sum_{i=1}^{p} \phi_i^{(t)} \circ X_{t-i} + \varepsilon_t \right)^m
= \sum_{k=0}^{m-1} C_m^k \sum_{i=1}^{p} (\phi_i^{(t)} \circ X_{t-i})^{m-k} \varepsilon_t^k + \varepsilon_t^m + R_m, \quad m = 1, 2, \ldots, 2p,
\]

(A.1)

where \(C_m^k\) is the number of combinations of size \(k\) from 1, 2, \ldots, \(m\) and

\[
R_m = X_t^m - \sum_{k=0}^{m-1} C_m^k \sum_{i=1}^{p} (\phi_i^{(t)} \circ X_{t-i})^{m-k} \varepsilon_t^k - \varepsilon_t^m.
\]

According to \(\{\varepsilon_s, s \geq t\}\) and \(\{\phi_i^{(s)}, i = 1, 2, \ldots, p; s \geq t\}\) are independent of \(\{X_s, s \leq t-1\}\), we have \(E(R_m|F_{t-1}) = 0\) from (2.1) and (2.2). Then

\[
E(X_t^m|F_{t-1}, J_{tp})
= E \left( \sum_{k=0}^{m-1} C_m^k \sum_{i=1}^{p} (\phi_i^{(t)} \circ X_{t-i})^{m-k} \varepsilon_t^k + \varepsilon_t^m \right| F_{t-1}, J_{tp})
= \sum_{k=0}^{m-1} C_m^k \sum_{i=1}^{p} E \left( (\phi_i^{(t)} \circ X_{t-i})^{m-k} | F_{t-1}, J_{tp} \right) E(\varepsilon_t^k) + E(\varepsilon_t^m).
\]

(A.2)
Let $S_i = \phi_i^{(t)} \circ X_{t-i}$, we know that $S_i$ is a conditional binomial distribution. Denote

$$S_i|\mathcal{F}_{t-1}, \mathcal{J}_t \sim B(X_{t-i}, \phi_i^{(t)}).$$

Then the conditional expected recursion formula of $S_i$ is as follows

$$E(S_{i+1}^k|\mathcal{F}_{t-1}, \mathcal{J}_t) = \phi_i^{(t)}(1-\phi_i^{(t)}) \frac{dE(S_i^k|\mathcal{F}_{t-1}, \mathcal{J}_t)}{d\phi_i^{(t)}} + X_{t-i} \phi_i^{(t)} E(S_i^k|\mathcal{F}_{t-1}, \mathcal{J}_t).$$

(A.3)

Since $\varepsilon_t$ has a poisson distribution with parameter $\lambda$, the quantities $E(\varepsilon_t^k)$ can be obtained by the expected recursion formula

$$E(\varepsilon_{t+1}^k) = \lambda \frac{dE(\varepsilon_t^k)}{d\lambda} + \lambda E(\varepsilon_t^k), \quad k = 1, 2, \ldots, m.$$  

(A.4)

Taking conditional expectations on both sides of (A.1), we obtain the following formula as follows

$$E(X_t^m|\mathcal{F}_{t-1}) = \sum_{k=0}^{m-1} C_m^k \sum_{i=1}^{p} E \left[ E(S_i^{m-k}|\mathcal{F}_{t-1}, \mathcal{J}_t)|\mathcal{F}_{t-1} \right] E(\varepsilon_t^k) + E(\varepsilon_t^m).$$

(A.5)

Next, substituting (A.3) and (A.4) into equation (A.5), we can obtain $E(X_t^m|\mathcal{F}_{t-1}), m = 1, 2, \ldots, 2p$.

Specifically, we have

$$E(X_t|\mathcal{F}_{t-1}) = \sum_{i=1}^{p} \alpha_i \phi_i X_{t-i} + \lambda,$$

$$\text{Var}(X_t|\mathcal{F}_{t-1}) = \sum_{i=1}^{p} \left[ \alpha_i \phi_i (1 - \phi_i) X_{t-i} + \alpha_i \phi_i^2 X_{t-i}^2 \right] + 2\lambda \sum_{i=1}^{p} \alpha_i \phi_i X_{t-i}$$

$$+ \lambda + \lambda^2 - \left( \sum_{i=1}^{p} \alpha_i \phi_i X_{t-i} + \lambda \right)^2,$$
Cov\( (X_t, X^2_t|\mathcal{F}_{t-1}) \)

\[
\begin{align*}
&= \sum_{i=1}^{p} \left[ \alpha_i \phi_i (1 - \phi_i)(1 - 2\phi_i)X_{t-i} + 3\alpha_i \phi^2_i (1 - \phi_i)X^2_{t-i} + \alpha_i \phi^3_i X^3_{t-i} \right] \\
&+ 3\lambda \sum_{i=1}^{p} [\alpha_i \phi_i (1 - \phi_i)X_{t-i} + \alpha_i \phi^2_i X^2_{t-i}] \\
&+ 3(\lambda + \lambda^2) \sum_{i=1}^{p} \alpha_i \phi_i X_{t-i} + \lambda + 3\lambda^2 + \lambda^3 \\
&- \left( \sum_{i=1}^{p} \alpha_i \phi_i X_{t-i} + \lambda \right) \left( \sum_{i=1}^{p} \left[ \alpha_i \phi_i (1 - \phi_i)X_{t-i} + \alpha_i \phi^2_i X^2_{t-i} \right] \right) \\
&- \left( \sum_{i=1}^{p} \alpha_i \phi_i X_{t-i} + \lambda \right) \left( 2\lambda \sum_{i=1}^{p} \alpha_i \phi_i X_{t-i} + \lambda + \lambda^2 \right).
\end{align*}
\]

Var\( (X^2_t|\mathcal{F}_{t-1}) \)

\[
\begin{align*}
&= \sum_{i=1}^{p} \left[ \alpha_i \phi_i (1 - \phi_i)(1 - 6\phi_i + 6\phi^2_i)X_{t-i} + \alpha_i \phi^2_i (1 - \phi_i)(7 - 11\phi_i)X^2_{t-i} \right] \\
&+ \sum_{i=1}^{p} \left[ 6\alpha_i \phi^3_i (1 - \phi_i)X^3_{t-i} + \alpha_i \phi^4_i X^4_{t-i} \right] \\
&+ 4\lambda \sum_{i=1}^{p} \left[ \alpha_i \phi_i (1 - \phi_i)(1 - 2\phi_i)X_{t-i} + 3\alpha_i \phi^2_i (1 - \phi_i)X^2_{t-i} + \alpha_i \phi^3_i X^3_{t-i} \right] \\
&+ 6(\lambda + \lambda^2) \sum_{i=1}^{p} \left[ \alpha_i \phi_i (1 - \phi_i)X_{t-i} + \alpha_i \phi^2_i X^2_{t-i} \right] \\
&+ 4(\lambda + 3\lambda^2 + \lambda^3) \sum_{i=1}^{p} \alpha_i \phi_i X_{t-i} + \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4 \\
&- \left( \sum_{i=1}^{p} \left[ \alpha_i \phi_i (1 - \phi_i)X_{t-i} + \alpha_i \phi^2_i X^2_{t-i} \right] \right)^2.
\end{align*}
\]

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References


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