A Weak Version of
Bivariate Lack of Memory Property

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Abstract. We suggest a modification of the classical Marshall-Olkin's bivariate exponential distribution considering a possibility of a singularity contribution along arbitrary line through the origin. It serves as a base of a weak version of the bivariate lack of memory property, which might be both "aging" and "non-aging' depending on the additional inclination (skew) parameter. The corresponding copula is obtained and we establish its disagreement with Lancaster's phenomena. Characterizations and properties of the novel bivariate memory-less notion are obtained and its applications are discussed. We characterize related weak multivariate version. The weak bivariate lack of memory property implies restrictions on associated marginal distributions. Starting from pre-specified marginals we propose a procedure to build bivariate distributions possessing a weak bivariate lack of memory property and illustrate it by examples. We complement the methodology with closure properties of the new class. We finish with a discussion and suggest several problems for a future research.

1 Introduction and preliminaries

The classical bivariate lack of memory property (BLMP) has at least half century history since the seminal technical report of Marshall and Olkin (1966). Many textbooks use as a base and give a special attention to the BLMP and related bivariate exponential distribution exhibiting singularity along the main diagonal in $R_+^2 = [0, \infty) \times [0, \infty)$, see Barlow and Proschan (1981), Singpurwalla (2006), Balakrishnan and Lai (2009), Gupta et al. (2010), Cherubini et al. (2015), McNeil et al. (2015) among others. More than 2000 articles complement and extend Marshall-Olkin's bivariate exponential distribution, justifying advantages in analysis of various data sets from engineering, medicine, insurance, finance, biology, etc. Let us mention two very recent contributions only: Lin et al. (2016) and Brigo et al. (2016). The reader would find in Lin et al. (2016) interesting and new inves-

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tigations regarding the correlation coefficient and dependence structure of bivariate Marshall-Olkin's exponential distribution. On the other side, Brigo et al. (2016) offer a new characterization of Marshall-Olkin multivariate law (e.g., all sub-vectors of associated survival indicators are continuous-time Markov chains) with a direct impact to overcome practical limitations for the modeling of high-dimensional default times. In this article we suggest one more modification of the BLMP, allowing singularity along the arbitrary line through the origin $(0, 0)$.

To proceed, let us begin with the classical bivariate Marshall-Olkin (MO) shock model specified by the stochastic representation

$$(X_1, X_2) = (\min(T_1, T_3), \min(T_2, T_3)), \tag{1.1}$$

where non-negative continuous random variables T_1 and T_2 identify the occurrence of independent "individual shocks" affecting two devices and T_3 is their "common shock". The random vector (X_1, X_2) presents the joint distribution of both lifetimes.

Specifically, let (X_1, X_2) be a continuous non-negative random vector defined by its joint survival function $S_{X_1, X_2}(x_1, x_2) = P(X_1 > x_1, X_2 > x_2)$ for all $x_1, x_2 \geq 0$. If the shocks are governed by independent homogeneous Poisson processes, then T_i's in (1.1) are exponentially distributed with parameters $\gamma_i > 0$, $i = 1, 2, 3$, and we obtain the *MO bivariate exponential distribution* given by

$$S_{X_1, X_2}(x_1, x_2) = \exp\{-\gamma_1 x_1 - \gamma_2 x_2 - \gamma_3 \max(x_1, x_2)\}, \quad x_1, x_2 \geq 0, \tag{1.2}$$

see Marshall and Olkin (1967). Due to the "common shock" distinguished by T_3 in (1.1), this distribution owns a singular component along the line $\{x_1 = x_2\}$ in R_+^2, with a weight $P(X_1 = X_2) = \frac{\gamma_3}{\gamma_1 + \gamma_2 + \gamma_3} > 0$, as shows the scatter plot displayed on Figure 1 (a). Hence, the MO bivariate exponential distribution is not absolutely continuous, i.e., it does not have a probability density with respect to the two-dimensional Lebesgue measure.

To give a probability formalization of model (1.1), consider a system composed by two items, to be denoted by 1 and 2. We associate with each item j a Bernoulli random variable Z_j, indicating whether the item is operational ($Z_j = 1$) or failed ($Z_j = 0$), $j = 1, 2$. The bivariate Bernoulli random vector (Z_1, Z_2) represents the state of the system. It is specified in terms of MO construction (1.1). The vector (X_1, X_2) exhibits the latent state of the system, since the MO model (1.2) is defined in terms of vector (T_1, T_2, T_3) of latent variables that identify independent exponential shock times. Each shock takes down a given subset of items ($\{1\}$, $\{2\}$ or $\{$both 1 and 2$\}$) and occurs at an exponential time with constant rates γ_1, γ_2 and γ_3, respectively.

The distribution (1.2) is the only solution with exponential marginals of functional equation

$$S_{X_1,X_2}(x_1 + t, x_2 + t) = S_{X_1,X_2}(x_1, x_2) S_{X_1,X_2}(t, t) \tag{1.3}$$

for all $x_1, x_2, t \geq 0$, characterizing the BLMP. Relation (1.3) tells us that, independent of t, the BLMP preserves the joint distribution of both (X_1, X_2) and its residual lifetime vector $\mathbf{X}_t = [(X_1 - t, X_2 - t) \mid X_1 > t, X_2 > t]$ and their marginal distributions, therefore.

The MO bivariate exponential distribution (1.2) has exponential marginals with parameters $\gamma_1 + \gamma_3$ and $\gamma_2 + \gamma_3$ and hence, constant marginal failure (hazard) rates. This restricts its usefulness for practical needs. As a response, other solutions of (1.3) with non-exponential marginals have been introduced. Let us mention Block and Basu (1974), Proschan and Sullo (1974), Friday and Patil (1977). An important contribution to the bivariate lack of memory notion is offered by Kulkarni (2006) who suggested a class of bivariate distributions possessing BLMP specified by (1.3), having increasing or/and decreasing marginal failure rates, but they should satisfy a set of restrictions. Many other authors are cited by Balakrishnan and Lai (2009) in their Chapter 10.

The stochastic relation (1.1) is widely used in literature. For example, Li and Pellerey (2011) launched the *Generalized MO model* considering non-exponential independent random variables T_i in (1.1), $i = 1, 2, 3$. The corresponding joint distributions do not possess BLMP, i.e., are "aging". As a further step, Pinto and Kolev (2015) introduced the *Extended MO model* assuming dependence between variables T_1 and T_2, but keeping T_3 independent of them. The motivation is that the individual shocks might be dependent if the items share a common environment. In this case however, BLMP may be fulfilled or not depending on parameters of joint distribution of (T_1, T_2) and distribution of T_3.

For a specific data set, it may happen that a cluster of the scatter plot points is concentrated on an arbitrary continuous curve in R_+^2, or along arbitrary line through the origin (0,0), in particular, as shown on Figure 1 (b). One would be unsuccessful to model adequately such bivariate data with distributions generated by stochastic representation (1.1).

A possible alternative to (1.2) is the *modified MO bivariate exponential distribution* defined as follows

$$S_{X_1,X_2}(x_1, x_2) = \exp\left\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, \omega x_2)\right\}, \tag{1.4}$$

where $\omega > 0$ and $x_1, x_2 \geq 0$. At least one of parameters λ_i, $i = 1, 2, 3$, should be positive, otherwise one would have a degenerate distribution. The

marginals in (1.4) are exponentially distributed with parameters $\lambda_1 + \lambda_3$ and $\lambda_2 + \omega\lambda_3$. One can recognize that there exists a positive mass concentrated along the line $L_\omega : \{x_1 = \omega x_2\}$ through the origin $(0,0)$ in R_+^2. It can be verified that the singularity contribution is $P(X_1 = \omega X_2) = \frac{\omega\lambda_3}{\omega\lambda_1 + \lambda_2 + \omega\lambda_3}$. Obviously, X_1 and X_2 are independent if $\lambda_3 = 0$ in (1.4).

Statistical inference related to the modified MO bivariate exponential distribution (1.4) is presented by Okyere (2007) who provided maximum likelihood estimation of parameters when ω is known, along with several asymptotic results.
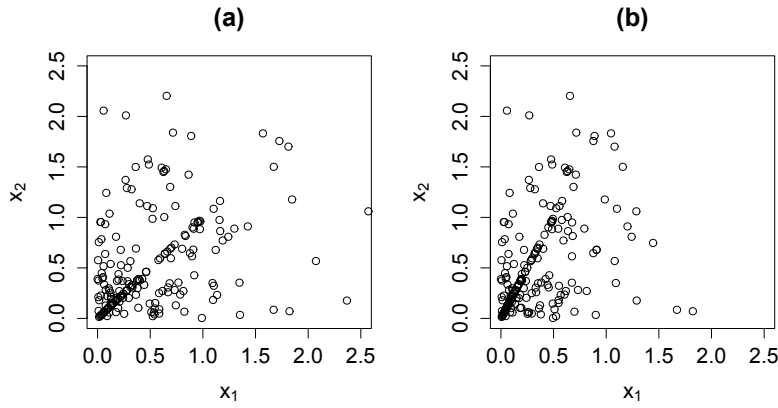

Figure 1: Plots with singularity along the lines $\{x_1 = x_2\}$ and $\{x_1 = 0.5x_2\}$.
(a) MO bivariate exponential distribution (1.2): $\omega = 1, \gamma_1 = \gamma_2 = \gamma_3 = 1$;
(b) Modified MO biv. exp. distribution (1.4): $\omega = 0.5, \lambda_1 = \lambda_2 = \lambda_3 = 1$.

One can check that (1.4) is a solution of the functional equation

$$S_{X_1,X_2}(x_1 + \omega t, x_2 + t) = S_{X_1,X_2}(x_1, x_2)\, S_{X_1,X_2}(\omega t, t), \qquad (1.5)$$

for all $x_1, x_2, t \geq 0$ and $\omega > 0$.

The BLMP functional equation (1.3) can be obtained from (1.5) when $\omega = 1$. In other words, the functional equation (1.5) exhibits a weak version of BLMP. Thus, we are ready to introduce the following

Definition 1.1. *A bivariate non-negative continuous distribution whose joint survival function satisfies the functional equation (1.5) possesses the* **weak bivariate lack of memory property***, or shortly W-BLMP.*

Our aim in this article is to investigate the W-BLMP defined by functional equation (1.5). In Section 2 we discuss structural properties of mod-

ified MO bivariate exponential distribution (1.4) and compare it with the classical one (1.2). We obtain the corresponding modified MO survival copula and discuss its features. In Section 3 we characterize the W-BLMP and present its multivariate version. In Section 4 we recognize that only certain marginal distributions are allowed for bivariate continuous distributions satisfying (1.5). The corresponding restrictions in terms of marginal densities and failure rates are derived. We suggest a procedure to construct bivariate distributions belonging to the W-BLMP class starting from given (pre-specified) marginal distributions. We support the methodology by typical examples and complement it by closure properties of the W-BLMP class. We finish with a discussion and pose several possible problems for further investigation.

2 Modified Marshall-Olkin bivariate exponential distribution

In fact, the modified MO bivariate exponential distribution (1.4) has been introduced by Esary and Marshall (1974) in their Example 2.3 by the following joint survival function

$$S_{X_1,X_2}(x_1,x_2) = \exp\left\{-\xi_1 x_1 - \xi_2 x_2 - \max\left(\xi_3 x_1, \xi_4 x_2\right)\right\}, \quad x_1, x_2 \geq 0,$$

where $\xi_1 + \xi_3 > 0$, $\xi_2 + \xi_4 > 0$ and $\xi_i \geq 0$, $i = 1, 2, 3, 4$. This distribution has been used as a base and motivation to define a novel class of bivariate distributions exhibiting exponential scaled minima. One can recognize the associated relation in the second proof of Theorem 1. If substitute $\lambda_i = \xi_i$, $i = 1, 2, 3$ and $\omega = \frac{\xi_4}{\xi_3}$ in last relation one will get (1.4).

Let us note that the modified MO bivariate exponential distribution (1.4) is positive quadrant dependent since $S_{X_1,X_2}(x_1,x_2) \geq S_{X_1}(x_1) S_{X_2}(x_2)$ for all $x_1, x_2 \geq 0$.

We will discuss in the next properties of the modified MO bivariate exponential distribution (1.4) and then we compare it with MO's distribution (1.2). We obtain the survival copula corresponding to (1.4) as well and outline the disagreement with Lancaster's phenomena.

2.1 Basic properties

The modified MO bivariate exponential distribution (1.4) can be generated by the following two stochastic representations:

1. Let (Y_1, Y_2) follow the MO bivariate exponential distribution (1.2) obtained by stochastic relation (1.1). The distribution (1.4) results if substitute $X_1 = Y_1$ and $X_2 = \frac{Y_2}{\omega}$ with $\omega > 0$ in (1.1). In this case, $\lambda_1 = \gamma_1$, $\lambda_2 = \omega\gamma_2$ and $\lambda_3 = \gamma_3$;

2. The joint survival function in (1.4) is a consequence of stochastic representation

$$(X_1, X_2) = \left[\min(T_1, T_3), \min\left(T_2, \frac{T_3}{\omega}\right) \right], \qquad (2.1)$$

where $\omega > 0$ and the random variables T_i are independent and exponentially distributed with parameters $\lambda_i > 0$, respectively, $i = 1, 2, 3$.

Really, for all $x_1, x_2 \geq 0$ and $\omega > 0$ we have

$$P(X_1 > x_1, X_2 > x_2) = P\left(\min(T_1, T_3) > x_1, \min\left(T_2, \frac{T_3}{\omega}\right) > x_2\right)$$
$$= P(T_1 > x_1, T_2 > x_2, T_3 > \max(x_1, \omega x_2)).$$

Taking into account that T_i are independent and exponentially distributed with parameters λ_i, $i = 1, 2, 3$, we obtain the joint survival function given by (1.4).

Remark 2.1. Reliability interpretation of (2.1). *Denote by X_i the lifetime of a component i, $i = 1, 2$. If $0 < \omega < 1$, relation (2.1) tells us that a common "fatal shock" destroys immediately the first component and has a delayed effect on the second one. For example, assume that only one of two identical devices operating in the same factory is equipped with a electricity generator protecting against possible blackout. The stochastic relation (2.1) indicates that the lifetime represented by X_2 will be greater than the lifetime X_1 of unprotected device if a common "fatal shock" (governed by a homogeneous Poisson process) occurs.*

Therefore, an important characteristic of distribution (1.4) resulting from (2.1), is that it permits a "late" failure of one component when a "fatal shock" strikes both components (as against to the MO models generated by (1.1), where both components fail simultaneously if occurs a common "fatal shock" distinguished by the random variable T_3).

On the other side, the physical meaning of equation (1.5) is the following: the conditional probability that both components survive an "additional" time (x_1, x_2) after surviving $(\omega t, t)$ time units is supposed to be equal to unconditional probability of surviving to time (x_1, x_2), i.e.,

$$P(X_1 > x_1 + \omega t, X_2 > x_2 + t \mid X_1 > \omega t, X_2 > t) = P(X_1 > x_1, X_2 > x_2).$$

In terms of joint survival function the last relation can be equivalently written by functional equation (1.5). It says that the (X_1, X_2) and corresponding ω-*type residual lifetime vector*, to denote it by

$$\mathbf{X}_{\omega, t} = [(X_1 - \omega t, X_2 - t) \mid X_1 > \omega t, X_2 > t],$$

should have the same joint distribution, independent of $t > 0$ for all $x_1, x_2 \geq 0$ and $\omega > 0$. The identical memory-less property is valid for marginal distributions of (X_1, X_2) and $\mathbf{X}_{\omega,t}$. Obviously, if $\omega = 1$, then $\mathbf{X}_{\omega,t}$ transforms into the residual lifetime vector $\mathbf{X}_t$ associated to the BLMP.

A basic issue in the bivariate data analysis is to study the monotonicity (e.g. increasing/decreasing performance) of the survival function of residual lifetime vector $\mathbf{X}_t$ corresponding to the vector (X_1, X_2). In fact, the main interest is to establish when the ratio $Q_1(x_1, x_2, t) = \frac{S_{X_1, X_2}(x_1+t, x_2+t)}{S_{X_1, X_2}(x_1, x_2)}$ is non-decreasing (non-increasing) in $x_1, x_2 \geq 0$ for all $t \geq 0$, see Barlow and Proschan (1981).

Therefore, an important problem would be to investigate the monotonicity of the ω-type residual lifetime vector $\mathbf{X}_{\omega,t}$ corresponding to the vector (X_1, X_2) as well. In this case the analysis is related to function $Q_2(x_1, x_2, t) = \frac{S_{X_1, X_2}(x_1+\omega t, x_2+t)}{S_{X_1, X_2}(x_1, x_2)}$, where $\omega > 0$.

Under the above notations, we establish relationships between bivariate exponential distributions (1.4) and (1.2) in the next two Lemmas.

Lemma 2.1. *Let (X_1, X_2) follow the modified MO bivariate exponential distribution (1.4) for some $\omega \in (0, 1)$. Then*

$$Q_1(x_1, x_2, t) = \begin{cases} S_{X_1, X_2}(t, t) = S_{X_3}(a_1 t), & \text{if } x_1 \geq \omega x_2, \ t \geq 0, \\ S_{X_3}(h(x_1, x_2, t)), & \text{if } x_1 < \omega x_2, \ t > \frac{\omega x_2 - x_1}{1-\omega}, \\ S_{X_3}(a_2 t), & \text{if } x_1 < \omega x_2, \ 0 \leq t \leq \frac{\omega x_2 - x_1}{1-\omega}, \end{cases}$$
(2.2)

where S_{X_3} is the survival function of unit exponential random variable X_3, $a_1 = \lambda_1 + \lambda_2 + \lambda_3$, $a_2 = \lambda_1 + \lambda_2 + \omega \lambda_3$ and $h(x_1, x_2, t) = a_1 t - \lambda_3(\omega x_2 - x_1) \geq 0$.

Proof. First note that the ratio $Q_1(x_1, x_2, t) = \frac{S_{X_1, X_2}(x_1+t, x_2+t)}{S_{X_1, X_2}(x_1, x_2)}$ represents the conditional probability

$$P_1(t \,|\, x_1, x_2) = P(X_1 > x_1 + t, X_2 > x_2 + t \mid X_1 > x_1, X_2 > x_2).$$

Let $x_1 \geq \omega x_2$. Apply (1.4) with $\omega \in (0, 1)$ to get the first expression for $Q_1(x_1, x_2, t)$ in (2.2).

Now assume that $x_1 < \omega x_2$ and use again (1.4) to obtain

$$S_{X_1, X_2}(x_1+t, x_2+t) = \exp\{-\lambda_1(x_1+t) - \lambda_2(x_2+t) - \lambda_3 \max[x_1+t, \omega(x_2+t)]\}.$$

If $t > \frac{\omega x_2 - x_1}{1-\omega}$, then $\max[x_1 + t, \omega(x_2 + t)] = x_1 + t$ and we arrive to the second expression in (2.2). In this case we have $\exp\{-h(x_1, x_2, t)\} \leq 1$, which ensures that conditional probability $P_1(t \,|\, x_1, x_2) \leq 1$.

When $t < \frac{\omega x_2 - x_1}{1-\omega}$ we have $\max[x_1 + t, \omega(x_2 + t)] = \omega(x_2 + t)$ and the third relation in (2.2) follows. $\qquad\square$

A graphical illustration of the ratio $\frac{S_{X_1,X_2}(x_1+t,x_2+t)}{S_{X_1,X_2}(x_1,x_2)}$ for fixed t and parameter values $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \omega = 0.5$ in (2.2) is given on Figure 2.
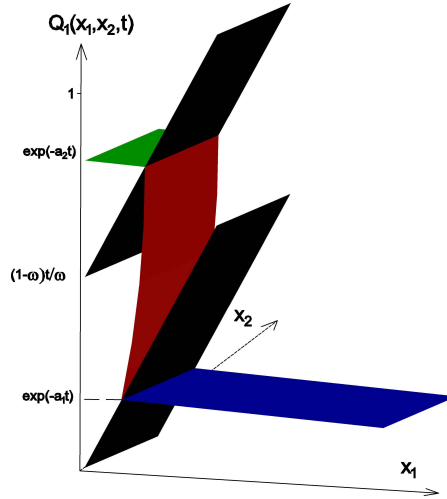

Figure 2: Graph of $Q_1(x_1, x_2, t) = \frac{S_{X_1,X_2}(x_1+t,x_2+t)}{S_{X_1,X_2}(x_1,x_2)}$ for a fixed t letting $\lambda_1 = 1.0, \lambda_2 = 1.0, \lambda_3 = 1.0$ and $\omega = 0.5$ in (2.2).

In Figure 3 we display the "level curves" of $Q_1(x_1, x_2, t)$ from Figure 2 by additionally fixing $x_2 = 0.76$ in (2.2). Observe that W-BLMP is represented by BLMP when $x_1 \geq \omega x_2$.

Remark 2.2. Graphical interpretation of (2.2). *The expressions in Lemma 2.1 can be interpreted as follows.*

1. The bivariate distribution (1.4) possesses BLMP below the line $\{x_2 = \frac{x_1}{\omega}\}$, i.e. the BLMP and W-BLMP worlds are equivalent given that $x_1 \geq \omega x_2$ and $\omega \in (0,1)$;

2. If $x_1 < \omega x_2$ and $t > \frac{\omega x_2 - x_1}{1-\omega}$, then the conditional probability $P_1(t \mid x_1, x_2)$ exhibits an "aging" behavior in the set $\{(x_1, x_2) \in R_+^2 : \frac{x_1}{\omega} < x_2 < \frac{x_1 + (1-\omega)t}{\omega}\}$.
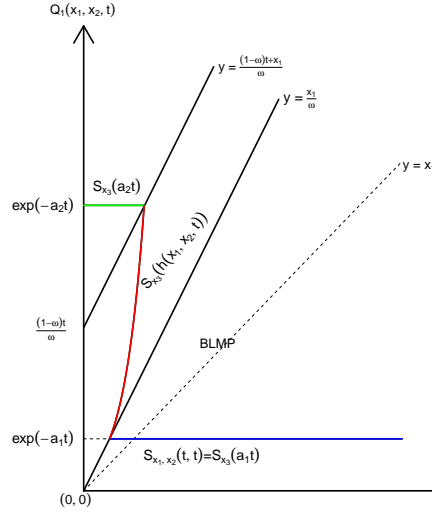
Figure 3: Level curves of $Q_1(x_1, x_2, t) = \frac{S_{X_1,X_2}(x_1+t, x_2+t)}{S_{X_1,X_2}(x_1,x_2)}$ for fixed t, when $x_2 = 0.76$, $\lambda_1 = 1.0, \lambda_2 = 1.0, \lambda_3 = 1.0$ and $\omega = 0.5$ in (2.2).

In this case, we can write

$$S_{X_1,X_2}(x_1 + t, x_2 + t) = S_{X_1,X_2}(x_1, x_2)S_{X_3}(h(x_1, x_2, t)),$$

where X_3 is exponentially distributed with parameter 1 evaluated at

$$h(x_1, x_2, t) = (\lambda_1 + \lambda_2 + \lambda_3)t - \lambda_3(\omega x_2 - x_1) \geq 0.$$

In this case W-BLMP is "aging";

3. If $x_1 < \omega x_2$ and $t \leq \frac{\omega x_2 - x_1}{1-\omega}$, then $P_1(t \,|\, x_1, x_2)$ is independent of x_1 and x_2 in the set $\{(x_1, x_2) \in R_+^2 : \frac{x_1 + (1-\omega)t}{\omega} \leq x_2 \leq \exp(-a_2 t)\}$, i.e., W-BLMP exhibits "non-aging" performance as well, but with a lower intensity than the usual one, since $a_2 < a_1$.

The function $Q_2(x_1, x_2, t) = \frac{S_{X_1,X_2}(x_1+\omega t, x_2+t)}{S_{X_1,X_2}(x_1,x_2)}$ is analyzed in the following claim. Note that $Q_2(x_1, x_2, t)$ is equal to the conditional probability

$$P_2(t \,|\, x_1, x_2) = P(X_1 > x_1 + \omega t, X_2 > x_2 + t \mid X_1 > x_1, X_2 > x_2).$$

Lemma 2.2. *Let (X_1, X_2) follow the MO bivariate exponential distribution*

(1.2). If $\omega \in (0,1)$ we have

$$Q_2(x_1, x_2, t) = \begin{cases} S_{X_1,X_2}(\omega t, t) = S_{X_3}(b_1 t), & \text{if } x_1 \leq x_2, \ t \geq 0, \\ S_{X_3}(g(x_1, x_2, t)), & \text{if } x_1 > x_2, \ t > \frac{x_1 - x_2}{1 - \omega}, \\ S_{X_3}(b_2 t), & \text{if } x_1 > x_2, \ 0 \leq t \leq \frac{x_1 - x_2}{1 - \omega}, \end{cases}$$

where S_{X_3} is the survival function of unit exponential random variable X_3, $b_1 = \gamma_1 \omega + \gamma_2 + \gamma_3$, $b_2 = \gamma_1 \omega + \gamma_2 + \gamma_3 \omega$ and $g(x_1, x_2, t) = b_1 t - \gamma_3(x_1 - x_2) \geq 0$.

Proof. Follows step by step the proof of Lemma 2.1.   □

A graphical interpretation of the expressions in Lemma 2.2 is given on Figure 4 for fixed values of the parameters and arguments x_2 and t in (1.2).
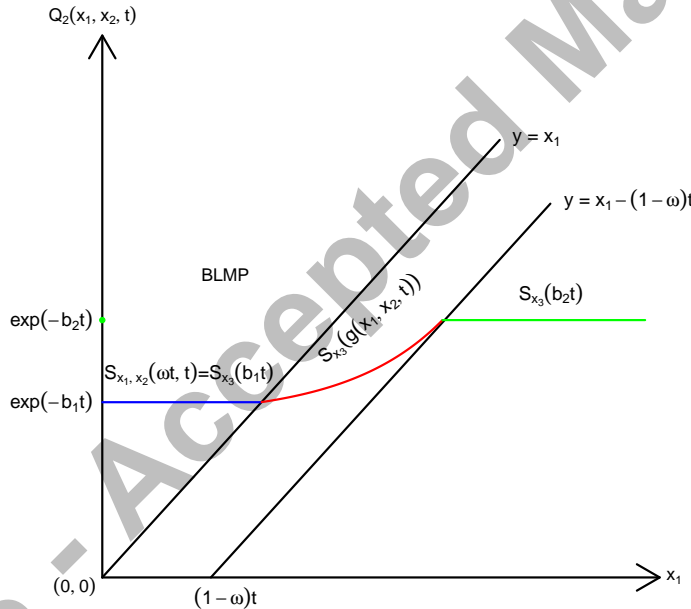

Figure 4: Level curves of $Q_2(x_1, x_2, t) = \frac{S_{X_1,X_2}(x_1 + \omega t, x_2 + t)}{S_{X_1,X_2}(x_1, x_2)}$ for fixed arguments $x_2 = 2.0$ and t, letting $\gamma_1 = 1, \gamma_2 = 1, \gamma_3 = 1$ and $\omega = 0.5$ in (1.2).

One can observe that the BLMP coincides with the W-BLMP in the set $\{x_2 \geq x_1\}$ and does not depend on x_1 and x_2 when $x_1 \geq x_2 - (1 - \omega)t$, i.e. being a "non-aging" and again with lower intensity than expected because

$b_1 > b_2$. But in the set $\{x_1 < x_2 < x_1 + (1 - \omega)t\}$ the W-BLMP exhibits an "aging" performance.

The expressions in Lemmas 2.1 and 2.2 are valid if $\omega \in (0, 1)$. One can obtain alternative formulas for $Q_1(x_1, x_2, t)$ and $Q_2(x_1, x_2, t)$ when $\omega > 1$.

2.2 Modified Marshall-Olkin survival copula

Denote by $\overline{C}_{MO}(u, v; \sigma_1, \sigma_2)$ the survival copula corresponding to MO bivariate exponential distribution (1.2). It is given by

$$\overline{C}_{MO}(u, v; \sigma_1, \sigma_2) = uv \min\{u^{-\sigma_1}, v^{-\sigma_2}\}, \quad u, v \in [0, 1],$$

where

$$\sigma_1 = \frac{\lambda_3}{\lambda_1 + \lambda_3} \quad \text{and} \quad \sigma_2 = \frac{\lambda_3}{\lambda_2 + \lambda_3}, \quad \text{with} \quad \lambda_i = \gamma_i, \, i = 1, 2, 3,$$

see Joe (2015), p. 183.

Now we will obtain the modified MO survival copula corresponding to (1.4). Let $S_{X_1, X_2}(x_1, x_2)$ be given by (1.4). The univariate survival functions are $S_{X_1}(x_1) = \exp\{-(\lambda_1 + \lambda_3)x_1\}$ and $S_{X_2}(x_2) = \exp\{-(\lambda_2 + \omega\lambda_3)x_2\}$. Since $\max(x_1, \omega x_2) = x_1 + \omega x_2 - \min(x_1, \omega x_2)$ we have

$$
\begin{aligned}
S_{X_1, X_2}(x_1, x_2) &= \exp\{-(\lambda_1 + \lambda_3)x_1 - (\lambda_2 + \omega\lambda_3)x_2 + \min(x_1, \omega x_2)\} \\
&= S_{X_1}(x_1) S_{X_2}(x_2) \exp\{\lambda_3 \min(x_1, \omega x_2)\} \\
&= S_{X_1}(x_1) S_{X_2}(x_2) \min[\exp\{\lambda_3 x_1\}, \exp\{\omega\lambda_3 x_2\}].
\end{aligned}
$$

Let $u = S_{X_1}(x_1)$, i.e., $x_1 = S_{X_1}^{-1}(u)$ and therefore $\exp\{\lambda_3 x_1\} = u^{-\frac{\lambda_3}{\lambda_1 + \lambda_3}}$. By analogy, putting $v = S_{X_2}(x_2)$ we obtain $\exp\{\omega\lambda_3 x_2\} = v^{-\frac{\omega\lambda_3}{\lambda_2 + \omega\lambda_3}}$. Thus, the *modified MO survival copula*, to denote it by $\overline{C}_{MMO}(u, v; \sigma_1, \phi_2(\omega))$, is given by

$$\overline{C}_{MMO}(u, v; \sigma_1, \phi_2(\omega)) = S_{X_1, X_2}\left(S_{X_1}^{-1}(u), S_{X_2}^{-1}(v)\right) = uv \min\{u^{-\sigma_1}, v^{-\phi_2(\omega)}\},$$

where $u, v \in (0, 1)$ and $\phi_2(\omega) = \frac{\omega\lambda_3}{\lambda_2 + \omega\lambda_3}$. This expression can be rewritten as

$$\overline{C}_{MMO}(u, v; \sigma_1, \phi_2(\omega)) = u^{\sigma_1} v^{\phi_2(\omega)} \min\{u^{1-\sigma_1}, v^{1-\phi_2(\omega)}\}. \tag{2.3}$$

On Figure 5 are displayed scatter plots of modified MO survival copula (2.3) with $\lambda_1 = 1.0, \lambda_2 = 0.3$ and $\lambda_3 = 1.0$ for four different values of ω. When $\omega = 1$ we recover the MO survival copula $\overline{C}_{MO}$, see Figure 5 (b). One can observe the influence of the skew parameter ω on the shape of the corresponding singularity line.

Several particular cases and properties of the modified MO survival copula (2.3) are listed below.
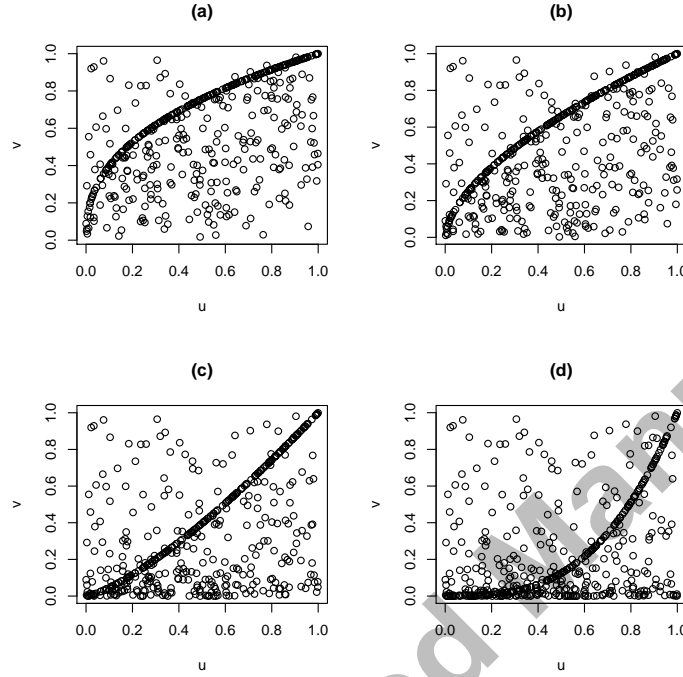
Figure 5: Scatter plots of modified MO copula for $\lambda_1 = 1.0, \lambda_2 = 0.3$ and $\lambda_3 = 1.0$ when (a) $\omega = 1.5$; (b) $\omega = 1.0$; (c) $\omega = 0.5$ and (d) $\omega = 0.2$.

- The MO survival copula $\overline{C}_{MO}(u, v; \sigma_1, \sigma_2)$ results if substitute $\omega = 1$ in (2.3);
- If put $\lambda_2 = \omega \lambda_2^*$ in (2.3) one will get the MO survival copula with a second parameter $\sigma_2^* = \frac{\lambda_3}{\lambda_2^* + \lambda_3}$, e.g., $\overline{C}_{MO}(u, v; \sigma_1, \sigma_2^*)$;
- When $\omega = 1$ and $\lambda_1 = \lambda_2$ or $\lambda_2 = \omega \lambda_1$ one obtains the symmetric Cuadras-Auge copula, see Joe (2015), p. 183;
- The lower and upper tail dependence coefficients $\lambda_L = \lim_{t \to 0^+} \frac{C(t,t)}{t}$ and $\lambda_U = \lim_{t \to 1^-} \frac{\overline{C}(t,t)}{1-t}$ of the modified MO copula are given by $\lambda_L = 0$ and $\lambda_U = \min\{\sigma_1, \phi_2(\omega)\}$. Thus, $\overline{C}_{MMO}(u, v; \sigma_1, \phi_2(\omega))$ can be used to model the extremal dependence as well.

Remark 2.3. Power representations of the modified MO copula.
First remind that if $C(u,v)$ is a copula, then its power copula $C_P(u,v)$ *is defined as*

$$C_P(u,v) = u^{\eta_1} v^{\eta_2} C(u^{1-\eta_1} v^{1-\eta_2})$$

for some additional parameters $\eta_1, \eta_2 \in [0,1]$. The following two relations

are valid.

- *From representation (2.3) one can recognize that the modified MO copula is, in fact, a power copula corresponding to the upper Frechet bound copula* $\min(u, v)$, *see Example 1 in Liebscher (2008);*
- *The MO survival copula* $\overline{C}_{MO}(u, v; \sigma_1, \sigma_2)$ *and its modified MO survival copula* $\overline{C}_{MMO}(u, v; \sigma_1, \phi_2(\omega))$ *are related by equation*

$$\overline{C}_{MMO}(u, v; \sigma_1, \phi_2(\omega)) = v^{1-\eta}\overline{C}_{MO}(u, v^{\eta}; \sigma_1, \sigma_2),$$

where the parameter $\eta = \frac{\omega(\lambda_2+\lambda_3)}{\lambda_2+\omega\lambda_3}$ *belongs to the interval* $[0, 1]$, *if and only if* $\omega \in [0, 1]$.

Therefore, the family of modified MO copulas can be used to model a complementary amount of bivariate asymmetry induced by $\overline{C}_{MO}(u, v; \sigma_1, \sigma_2)$, *see supporting comments in Joe (2015), p. 184. This additional asymmetry does not imply, in general, an increase of extremal dependence governed by the MO survival copula exhibiting upper tail dependence coefficient* $\lambda_U = \min\{\sigma_1, \sigma_2\}$. *Depending on parameter values it may happen that* $\phi_2(\omega) < \sigma_2$. *As a confirmation, compare the scatterplot (b) on Figure 5 (corresponding to MO survival copula) with others.*

The four parameters $\lambda_1, \lambda_2, \lambda_3$ and ω of modified MO model (1.4) are reduced to three parameters (e.g., σ_1 and $\phi_2(\omega)$) of the associated survival copula specified by (2.3). For corresponding Spearman's rho $\rho_C = 12 \int_0^1 \int_0^1 \overline{C}_{MMO}(u, v) du dv - 3$, we obtain

$$\rho_C = \frac{3\sigma_1\phi_2(\omega)}{2\sigma_1 - \sigma_1\phi_2(\omega) + 2\phi_2(\omega)}.$$

The last expression in terms of $\lambda_1, \lambda_2, \lambda_3$ and ω can be rewritten as

$$\rho_C = \frac{3\omega\lambda_3}{2(\omega\lambda_1 + \lambda_2) + 3\omega\lambda_3} = \frac{\omega\lambda_3}{\frac{2}{3}(\omega\lambda_1 + \lambda_2) + \omega\lambda_3}.$$

If $S_{X_1, X_2}(x_1, x_2)$ is specified by (1.4), the Pearson correlation coefficient ρ_{W-BLMP} is given by

$$\rho_{W-BLMP} = P(X_1 = \omega X_2) = \frac{\omega\lambda_3}{\omega\lambda_1 + \lambda_2 + \omega\lambda_3},$$

i.e. $\rho_{W-BLMP} < \rho_C$. Hence, we proved the following statement.

Lemma 2.3. *The Pearson's correlation coefficient of the modified MO bivariate exponential distribution (1.4) is less than the Spearman's rho of the associated modified MO copula (2.3), regardless of parameters.*

Since a nonlinear transformation of bivariate continuous distribution results in associated copula, Lemma 2.3 implies that distribution (1.4) contradicts the Lancaster's phenomena stating that any nonlinear transformation of variables decreases the correlation in absolute value, see Lancaster (1957).

Note that $\rho_{\text{W-BLMP}}$ increases as λ_3 increases while keeping $\omega\lambda_1 + \lambda_2$ fixed. Similarly, ρ_C decreases when $\omega\lambda_1 + \lambda_2$ increases while keeping λ_3 fixed. When $\omega = 1$ one will get the conclusions obtained by Lin et al. (2016) for MO distribution (1.2), where complementary properties of dependence structure are obtained.

3 Characterizations

Here we will characterize the W-BLMP offering two alternative proofs. A multivariate version of W-BLMP will be presented as well.

3.1 Characterization of the weak bivariate lack of memory property

To proceed, denote by $S_X(x) = P(X > x)$ the survival function of continuous random variable X. Let us recall that the exponential distribution with a parameter $\delta > 0$ is defined by $S_X(x) = \exp\{-\delta x\}$, $x \geq 0$. It is characterized by a functional equation

$$S_X(x + t) = S_X(x)S_X(t) \quad \text{for all} \quad x \geq 0 \quad \text{independent of} \quad t > 0,$$

postulating the classical univariate memory-less property. An immediate bivariate extension of the former relation is the functional equation

$$S_{X_1,X_2}(x_1 + t_1, x_2 + t_2) = S_{X_1,X_2}(x_1, x_2)\, S_{X_1,X_2}(t_1, t_2) \qquad (3.1)$$

in the set $\mathcal{M} = \{x_1, x_2, t_1, t_2 \geq 0\}$, representing the stronger version of bivariate lack of memory property. As a consequence of Theorem 1 in Aczél (1966), see pp. 215-216, its general solution is given by

$$S_{X_1,X_2}(x_1, x_2) = \exp\{-\delta_1 x_1 - \delta_2 x_2\}, \quad \text{for some} \quad \delta_1, \delta_2 > 0. \qquad (3.2)$$

Marshall and Olkin (1967) did an alternative proof of the same statement and conclude in their Lemma 2.1 that the solution (3.2) of (3.1) leads to independent exponential marginal distributions with parameters δ_1 and δ_2.

Remark 3.1. Two wrong counterexamples. *The BLMP specified by (1.3) is a particular case of (3.1) when $t_1 = t_2 = t$ and the solution (3.2)*

is the independent solution of (1.3) when $\gamma_3 = 0$ in (1.2). Below we present our "counterexamples".

1. One can verify that the modified MO bivariate exponential distribution (1.4) being a solution of (1.5) is also solution of functional equation (3.1) in the special case when $t_1 = \omega t$ and $t_2 = t$. Hence, in addition to the independent solution (3.2) of (3.1) we got non-independent exponential random variables with parameters $\lambda_1 + \lambda_3$ and $\lambda_2 + \omega\lambda_3$ whose joint distribution (1.4) satisfies (3.1) as well. So, the first lucky impression is that we got a counterexample. Formally, this is true (as in BLMP case, i.e., the joint survival function given by (1.2) is a solution of (3.1) when $t_1 = t_2 = t$). However, the reason of this wrong impression and "counterexample" is that the domain $\{x_1, x_2, t \geq 0 \text{ and } \omega > 0\}$ of functional equation (1.5) belongs to the domain $\mathcal{M}$ of (3.1). Therefore, not all solutions of (1.5) are solutions of (3.1). But, the independent solution (3.2) of functional equation (3.1) may serve as a particular solution of (1.5) and we will use this fact in the proof of Theorem 3.1. A nice related discussion can be found in Marshall and Olkin (1991);

2. Interestingly, the MO bivariate exponential distribution (1.2) is the only solution with exponential marginals of functional equation (3.1) for all nonnegative x_1, x_2, t_1, t_2, but belonging to the set

$$\{x_1 \leq x_2, \, t_1 \leq t_2\} \cup \{x_1 \geq x_2, \, t_1 \geq t_2\},$$

being a subset of $\mathcal{M}$, see Marshall and Shaked (1979).

We will obtain a general solution of functional equation (1.5) in the next statement, or equivalently, we will characterize the W-BLMP.

Theorem 3.1. *(i) The non-negative continuous random vector (X_1, X_2) possesses the W-BLMP defined by (1.5) if and only if its joint survival function is given by*

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp\{-\theta x_2\} S_{X_1}(x_1 - \omega x_2), & \text{if} \quad x_1 \geq \omega x_2, \\ \exp\{-\frac{\theta x_1}{\omega}\} S_{X_2}(x_2 - \frac{x_1}{\omega}), & \text{if} \quad x_1 \leq \omega x_2, \end{cases} \quad (3.3)$$

for all $x_1, x_2 \geq 0$, $\omega > 0$ and some $\theta > 0$, where $S_{X_i}(x_i)$ is the marginal survival function of X_i, $i = 1, 2$.

(ii) The only solution of (1.5) with exponential marginals is the modified MO bivariate exponential distribution (1.4).

Proof. Let (X_1, X_2) possess W-BLMP, or equivalently, let (1.5) be true.

Set $x_2 = 0$ in (1.5) to get $S_{X_1, X_2}(x_1 + \omega t, t) = S_{X_1}(x_1) S_{X_1, X_2}(\omega t, t)$. Substitute $x_1 = x_1 + \omega t$ and $x_2 = t$ to conclude that

$$S_{X_1, X_2}(x_1, x_2) = S_{X_1}(x_1 - \omega x_2) S_{X_1, X_2}(\omega x_2, x_2), \quad \text{if} \quad x_1 - \omega x_2 \geq 0. \quad (3.4)$$

As we noted in Remark 2.2, since the domain $\{x_1, x_2, t \geq 0 \text{ and } \omega > 0\}$ of (1.5) belongs to the domain $\mathcal{M} = \{x_1, x_2, t_1, t_2 \geq 0\}$ of functional equation (3.1), it follows that the solution (3.2) of (3.1) may serve as a particular solution of (1.5), i.e., it holds

$$S_{X_1,X_2}(\omega x_2, x_2) = \exp\{-(\alpha_1\omega + \alpha_2)x_2\} \quad \text{for some} \quad \alpha_1, \alpha_2 > 0.$$

Therefore, from (3.4) we obtain

$$S_{X_1,X_2}(x_1, x_2) = \exp\{-(\alpha_1\omega + \alpha_2)x_2\}S_{X_1}(x_1 - \omega x_2) \quad \text{if} \quad x_1 - \omega x_2 \geq 0,$$

which is the first relation in (3.3) letting $\theta = \alpha_1\omega + \alpha_2$.

By analogy, put $x_1 = 0$ in (1.5) to conclude that

$$S_{X_1,X_2}\left(x_1, \frac{x_1}{\omega}\right) = \exp\left\{-\frac{(\beta_1\omega + \beta_2)x_1}{\omega}\right\}$$

and to obtain the second expression in (3.3) with $\theta = \beta_1\omega + \beta_2$ for some $\beta_1 > 0$ and $\beta_1 > 0$ satisfying (3.2).

In order to ensure the same joint survival function $S_{X_1,X_2}(x_1, x_2)$ along the line $L_\omega : \{x_1 = \omega x_2\}$ one should require equality $\beta_1\omega + \beta_2 = \alpha_1\omega + \alpha_2$ for all possible values of parameters involved. Therefore, (3.3) is valid.

Conversely, it is direct to check that (3.3) satisfies the functional equation (1.5) and the part (i) of the statement is established.

To prove (ii), suppose that $S_{X_1,X_2}(x_1, x_2)$ is a survival function solving functional equation (1.5). Then Theorem 3.1 (i) implies its representation by (3.3). Assume that marginals of (X_1, X_2) are exponentially distributed. This means that $S_{X_1}(x_1) = \exp\{-\alpha_1 x_1\}$ and $S_{X_2}(x_2) = \exp\{-\beta_2 x_2\}$ for all $x_1, x_2 \geq 0$ and some $\alpha_1, \beta_2 > 0$. Replace these expressions in (3.3) to obtain

$$S_{X_1,X_2}(x_1, x_2) = \begin{cases} \exp\{-\alpha_1 x_1 - (\theta - \omega\alpha_1)x_2\}, & \text{if} \quad x_1 \geq \omega x_2, \\ \exp\{-\frac{(\theta - \beta_2)x_1}{\omega} - \beta_2 x_2\}, & \text{if} \quad x_1 \leq \omega x_2. \end{cases} \quad (3.5)$$

In (3.5) substitute

$$\omega\lambda_1 = \theta - \beta_2, \quad \lambda_2 = \theta - \omega\alpha_1 \quad \text{and} \quad \omega\lambda_3 = \omega\alpha_1 + \beta_2 - \theta. \quad (3.6)$$

Note that $S_{X_1,X_2}(x_1, x_2)$ given by (3.5) is a survival function being non increasing in its arguments. Therefore, inequalities $\theta - \beta_2 \geq 0$ and $\theta - \omega\alpha_1 \geq 0$ have to be satisfied, i.e., $\lambda_1, \lambda_2 \geq 0$ for $\omega > 0$ and hence, $\omega\alpha_1 + \beta_2 \leq 2\theta$.

We will show that the parameter λ_3 is non-negative as well. Following the brilliant method used by Gupta et al. (2010) in the proof of their Theorem

6.2, pp. 123-124, let $F_{X_1,X_2}(x_1, x_2)$ be the joint distribution of (X_1, X_2). Then the function

$$F(x_1) = F_{X_1,X_2}(\omega x_1, x_1) = 1 - S_{X_1}(\omega x_1) - S_{X_2}(x_1) + S_{X_1,X_2}(\omega x_1, x_1)$$

defines a proper continuous univariate distribution function with a density

$$f(x_1) = \frac{d}{dx_1} F(x_1) = \omega \alpha_1 \exp\{-\omega \alpha_1 x_1\} + \beta_2 \exp\{-\beta_2 x_1\} - \theta \exp\{-\theta x_1\} \geq 0.$$

Therefore, $\lim_{x_1 \to 0^+} f(x_1) = \omega \alpha_1 + \beta_2 - \theta \geq 0$, i.e. $\omega \alpha_1 + \beta_2 \geq \theta$.

Thus, the parameter space $\{\alpha_1, \beta_2, \theta, \omega > 0\}$ of $S_{X_1,X_2}(x_1, x_2)$ specified by (3.5) is restricted by the inequalities $\theta \leq \omega \alpha_1 + \beta_2 \leq 2\theta$.

Finally, solving the system (3.6) we obtain

$$\alpha_1 = \lambda_1 + \lambda_3, \quad \beta_2 = \lambda_2 + \omega \lambda_3 \quad \text{and} \quad \theta = \lambda_1 + \lambda_2 + \omega \lambda_3$$

and restore (1.4) via (3.5). Note that at least one of parameters λ_i, $i = 1, 2, 3$, should be positive, otherwise one would have a degenerate distribution. To finish the proof, it is easy to verify that (1.4) satisfies (1.5) indeed (as we did just before Remark 2.1). $\qquad\qquad\square$

Second proof of Theorem 3.1 (i). Instead of using a particular solution (3.2) of functional equation (1.5) in the proof of Theorem 3.1, one can proceed as follows.

Let (1.5) be true and set $x_2 = 0$, $y_1 = x_1 + \omega t$ and $y_2 = t$ to get relation (3.4) when $y_1 \geq \omega y_2$. Now, put $x_1 = \omega t$ and $x_2 = t$ in (3.4) to obtain $S_{X_1,X_2}(2\omega t, 2t) = [S_{X_1,X_2}(\omega t, t)]^2$ for all $t \geq 0$ and $\omega > 0$. Iterating the procedure one will arrive to the functional equation

$$S_{X_1,X_2}(k\omega t, kt) = [S_{X_1,X_2}(\omega t, t)]^k,$$

which is valid for all $k = 1, 2, ...$ and $t \geq 0$. Let $\Gamma(t) = S_{X_1,X_2}(k\omega t, kt)$ and the last equality becomes

$$\Gamma(t) = \left[\Gamma\left(\frac{t}{k}\right)\right]^k \quad \text{for all} \quad t > 0 \quad \text{and} \quad k = 1, 2, ...$$

This relation is a consequence of the functional equation $\Gamma(x+t) = \Gamma(x)\Gamma(t)$ for $x, t \geq 0$, characterizing the exponential distribution. Therefore, $\Gamma(t) = \exp\{-\theta t\}$ for some $\theta > 0$, see Section 5.3 in Galambos and Kotz (1978). Thus, we got the first expression in (3.3).

Using similar arguments one will obtain the second equation in (3.3) if $x_1 \leq \omega x_2$.

Remark 3.2. Independent case. *In the proof of Theorem 3.1 we used the condition $\beta_1\omega + \beta_2 = \alpha_1\omega + \alpha_2$ to ensure the same expression for $S_{X_1,X_2}(x_1, x_2)$ along the line $L_\omega : \{x_1 = \omega x_2\}$. If it happens that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, then the solution of the system (3.6) will indicate that $\lambda_3 = 0$, i.e., X_1 and X_2 are independent and exponentially distributed. Only in this case the general solution (3.2) of functional equation (3.1) is independent solution of (1.5) as well, see an alternative proof of Corollary 4.2.*

3.2 Multivariate weak lack of memory property

Let us comment briefly the multivariate case. Denote by $S_n(\mathbf{x})$ the survival function of nonnegative random vector $\mathbf{X} = (X_1, ..., X_n)$, where $\mathbf{x} = (x_1, ..., x_n) \in R_+^n$, $n \geq 2$.

The multivariate analog of functional equation (1.3) can be written as

$$S_n((\nu + 1)\mathbf{x}) = S_n(\nu\mathbf{x})S_n(\mathbf{x}) \quad \text{for all} \quad \nu > 0, \ \mathbf{x} \in R_+^n.$$

It turns out that former equation can be equivalently represented by

$$\left[S_n\left(\frac{t\mathbf{x}}{m}\right)\right]^m = S_n(t\mathbf{x}) \quad \text{for all} \quad t > 0, \ \mathbf{x} \in R_+^n \quad \text{and} \quad m = 1, 2, ...,$$

see Theorem 3.1.b in Marshall and Olkin (1991). Therefore, the vector $\mathbf{X}$ has a distribution with exponential scaled minima, i.e., $\min_{i \in I}\{\nu_i X_i\}$ has an exponential distribution for all $\nu_i > 0$, $i = 1, ..., n$ and all non-empty sets $I \in \{1, 2, ..., n\}$.

We will finish this section with a definition and characterization of a multivariate weak lack of memory property corresponding to functional equation (1.5). It is specified by relation

$$S_n(x_1 + x_i\omega, x_2 + x_i, ..., x_n + x_i) = S_n(x_1, x_2, ..., x_n) S_n(x_i\omega, x_i..., x_i) \ (3.7)$$

for $i = 1, 2, ..., n$.

Definition 3.1. *A random vector $\mathbf{X}$ is said to have a **weak multivariate lack of memory property** (to be abbreviated W-MLMP) if equation (3.7) holds.*

Let $R_n(\mathbf{x}) = -\ln S_n(\mathbf{x})$ and denote $\mathbf{1}_\omega = (1, 1, ..., \omega^{-1}, ..., 1)$, e.g., all elements of the vector $\mathbf{1}_\omega$ are equal to 1, instead of i-th which is ω^{-1}, $i = 1, 2, ..., n$. The following characterization of W-MLMP is true.

Theorem 3.2. *The vector $\mathbf{X}$ have W-MLMP if and only if there exist numbers $\theta > 0$ and $\omega > 0$ such that the following statements are equivalent for all $\mathbf{x} > \mathbf{0}$ and $t > 0$ with $\mathbf{x} - \mathbf{1}_\omega t \geq 0$:*

(i) $S_n(\mathbf{x}) = exp\left(-\frac{\theta t}{\omega}\right) S_n\left(\mathbf{x} - \mathbf{1}_\omega t\right);$

(ii) $R_n(\mathbf{x}) = \frac{\theta t}{\omega} + R_n\left(\mathbf{x} - \mathbf{1}_\omega t\right);$

(iii) $R_n\left(\mathbf{x} + x\mathbf{1}_\omega\right) = R_n(\mathbf{x}) + R_n\left(x\mathbf{1}_\omega\right).$

Proof. Suppose $\mathbf{X}$ have W-MLMP, i.e., equation (3.7) holds. Thus,

$$S_n(\mathbf{x}) = S_n\left(x_1 - t + t, ..., x_{i-1} - t + t, x_i - \frac{t}{\omega} + \frac{t}{\omega}, x_{i+1} - t + t, ..., x_n - t + t\right).$$

Applying (3.7) we get $S_n(\mathbf{x}) = S_n\left(\mathbf{x} - \mathbf{1}_\omega t\right) S_n\left(\mathbf{1}_\omega t\right).$

Taking $x_i = t$ for all $i = 1, ..., n$ in (3.7) and substituting $G(\omega, t) = S_n(\omega t, t, ..., t)$ we obtain $G(\omega + 1, t) = G(\omega, t)G(1, t)$ which can be reduced to multiplicative Cauchy functional equation with a solution $G(\omega, t) = \exp\{-c\omega t\}$ for arbitrary constant $c > 0$, see Aczél (1966), p. 35. Thus, $S\left(\mathbf{1}_\omega t\right) = \exp\left(-\frac{\theta}{\omega} t\right)$ and hence the relation (i) in the Theorem 3.2 is fulfilled.

The other equivalent implications are result of standard transformations (logarithmic and exponential), or can be checked trivially. □

Let us note that when $\omega = 1$ one will obtain the corresponding characterization of the multivariate lack of memory property established by Kulkarni (2006), see her Theorem 1.

4 Marginal restrictions

Theorem 3.1 characterizes bivariate continuous distributions in R_+^2 possessing the W-BLMP with joint survival function $S_{X_1,X_2}(x_1, x_2)$ specified by relation (3.3). It happens that the set of possible marginal distributions of bivariate continuous distributions possessing W-BLMP is restricted. In other words, the corresponding joint survival function is valid, e.g., $\frac{\partial^2 S_{X_1,X_2}(x_1,x_2)}{\partial x_1 \partial x_2} \geq 0$, only for certain marginal distributions of X_1 and X_2. Kulkarni (2006) did a similar conclusion about marginals of bivariate distributions possessing BLMP.

Here we will obtain the associated constraints in terms of marginal densities and hazard rates. As a result, we will get the admissible values of the parameter θ in the joint survival function given by (3.3). Thus one would be able to generate a wealth of bivariate distributions possessing W-BLMP having given marginals. We illustrate the methodology by several examples and establish related closure properties.

4.1 Marginal density restrictions

One can deduce that $S_{X_1,X_2}(x_1, x_2)$ specified by (3.3) admits the Lebesgue decomposition

$$S_{X_1,X_2}(x_1, x_2) = (1 - \alpha)\, S^{ac}_{X_1,X_2}(x_1, x_2) + \alpha S^{si}_{X_1,X_2}(x_1, x_2)\mathbf{I}_{\{x_1 = \omega x_2\}}, \quad (4.1)$$

where $\alpha = P(X_1 = \omega X_2) \geq 0$, $\mathbf{I}_{\{E\}}$ is the indicator function of the set E, $S^{si}_{X_1,X_2}(x_1, x_2)\mathbf{I}_{\{x_1 = \omega x_2\}}$ is a singular component with support along the line $L_\omega = \{x_1 = \omega x_2\}$ and $S^{ac}_{X_1,X_2}(x_1, x_2)$ is an absolutely continuous survival function. Hence, the joint continuous density $f_{X_1,X_2}(x_1, x_2)$ corresponding to (4.1) can be written as

$$f_{X_1,X_2}(x_1, x_2) = (1 - \alpha)\, f^{ac}_{X_1,X_2}(x_1, x_2) + \alpha f^{si}_{X_1,X_2}(x_1, x_2)\mathbf{I}_{\{x_1 = \omega x_2\}}.$$

Clearly, if $\alpha = P(X_1 = \omega X_2) = 0$, then the joint distribution of the random vector (X_1, X_2) in (3.3) will be absolutely continuous.

The next theorem shows a list of associated constraints in terms of marginal densities.

Theorem 4.1. *Let X_i be a non-negative random variable with absolutely continuous density $f_{X_i}(x_i)$ on all finite intervals, $i = 1, 2$. Then $S_{X_1,X_2}(x_1, x_2)$ specified by (3.3) is a proper bivariate survival function if only if*

$$\theta \leq \omega f_{X_1}(0) + f_{X_2}(0) \leq 2\theta \qquad (4.2)$$

and the following two inequalities are fulfilled

$$\theta + \omega \frac{d}{dx_1} \ln[f_{X_1}(x_1 - \omega x_2)] \geq 0, \quad for \quad x_1 \geq \omega x_2, \qquad (4.3a)$$

$$\theta + \frac{d}{dx_2} \ln\left[f_{X_2}\left(x_2 - \frac{x_1}{\omega}\right)\right] \geq 0, \quad for \quad x_1 \leq \omega x_2. \qquad (4.3b)$$

Proof. Since the survival function $S_{X_1,X_2}(x_1, x_2)$ may have a singular component along the line $L_\omega = \{x_1 = \omega x_2\}$, then $S_{X_1,X_2}(x_1, x_2)$ given by (3.3) will be proper if and only if both the absolutely continuous part $S^{ac}_{X_1,X_2}(x_1, x_2)$ and the singular part $S^{si}_{X_1,X_2}(x_1, x_2)\mathbf{I}_{\{x_1 = \omega x_2\}}$ are survival functions and satisfy (4.1) with a singularity weight $\alpha = P(X_1 = \omega X_2)$.

Starting from (3.3), after some algebra one will find that the function $(1 - \alpha)\, f^{ac}_{X_1,X_2}(x_1, x_2)$ can be written as

$$\begin{cases} \theta \exp(-\theta x_2) f_{X_1}(x_1 - \omega x_2) + \omega \exp(-\theta x_2)\frac{d}{dx_1} f_{X_1}(x_1 - \omega x_2), & \text{if } x_1 \geq \omega x_2, \\ \frac{\theta}{\omega} \exp\left(-\frac{\theta x_1}{\omega}\right) f_{X_2}\left(x_2 - \frac{x_1}{\omega}\right) + \frac{1}{\omega} \exp\left(-\frac{\theta x_1}{\omega}\right)\frac{d}{dx_2} f_{X_2}\left(x_2 - \frac{x_1}{\omega}\right), & \text{if } x_1 \leq \omega x_2. \end{cases} \quad (4.4)$$

By assumption, densities $f_{X_i}(x_i)$ are absolutely continuous on all finite intervals, $i = 1, 2$. This implies that the derivative $\frac{d}{dx_i} f_{X_i}(x_i)$ is integrable on $(-\infty, \infty)$, see Block and Basu (1974) for related discussion. So, we can compute

$$P\left(X_1 > \omega X_2\right) = \int_0^\infty \int_0^{\frac{x_1}{\omega}} \left(1 - \alpha\right) f_{X_1, X_2}^{ac}\left(x_1, x_2\right) dx_2 dx_1 = 1 - \frac{\omega f_{X_1}(0)}{\theta}$$

and

$$P\left(X_1 < \omega X_2\right) = \int_0^\infty \int_0^{\omega x_2} \left(1 - \alpha\right) f_{X_1, X_2}^{ac}\left(x_1, x_2\right) dx_1 dx_2 = 1 - \frac{f_{X_2}(0)}{\theta}.$$

But $\alpha = P\left(X_1 = \omega X_2\right) = 1 - P\left(X_1 > \omega X_2\right) - P\left(X_1 < \omega X_2\right)$ and we conclude that

$$\alpha = -1 + \frac{1}{\theta}\left[\omega f_{X_1}(0) + f_{X_2}(0)\right] \in [0, 1],$$

which implies inequalities (4.2).

Now, since $\frac{\partial^2 S_{X_1, X_2}(x_1, x_2)}{\partial x_1 \partial x_2}$ should be non-negative, from (4.4) we obtain

$$\exp\left(-\theta x_2\right)\left[\theta f_{X_1}\left(x_1 - \omega x_2\right) + \omega \frac{d}{dx_1} f_{X_1}\left(x_1 - \omega x_2\right)\right] \geq 0, \text{ if } x_1 \geq \omega x_2$$

and

$$\omega^{-1} \exp\left(-\frac{\theta x_1}{\omega}\right)\left[\theta f_{X_2}\left(x_2 - \frac{x_1}{\omega}\right) + \frac{d}{dx_2} f_{X_2}\left(x_2 - \frac{x_1}{\omega}\right)\right] \geq 0, \text{ if } x_1 \leq \omega x_2.$$

Taking into account that $f_{X_1}\left(x_1 - \omega x_2\right) \geq 0$ for $x_1 \geq \omega x_2$ and $f_{X_2}\left(x_2 - \frac{x_1}{\omega}\right) \geq 0$ when $x_1 \leq \omega x_2$, we arrive to inequalities (4.3a) and (4.3b). $\qquad\square$

Remark 4.1. (Exponential case). *In the proof of Theorem 3.1 (ii) we established that $\theta \leq \omega \alpha_1 + \beta_2 \leq 2\theta$, where α_1 and β_2 are parameters of exponentially distributed random variables X_1 and X_2, correspondingly. The last inequalities are particular case of restrictions (4.2) since $f_{X_1}(0) = \alpha_1$ and $f_{X_2}(0) = \beta_2$. In this case (4.3a) and (4.3b) are satisfied as well.*

Following the corresponding procedure used in the proof of Theorem 5.1 in Marshall and Olkin (1967) we get

Lemma 4.1. *The singular component of $S_{X_1, X_2}(x_1, x_2)$ specified by (3.3) is given by*

$$S_{X_1, X_2}^{si}(x_1, x_2)\mathbf{I}_{\{x_1 = \omega x_2\}} = \exp\left\{-\theta \max\left(x_1, \omega x_2\right)\right\}.$$

If set $\omega = 1$ in expression given by Lemma 4.1, one will get the singular component in BLMP case, see equation (5.6) in Marshall and Olkin (1967).

Notice that if inequalities (4.3a) and (4.3b) are fulfilled for some $\theta = \theta_0 > 0$, then they are also satisfied for all $\theta > \theta_0$. Denote by

$\tau = \{$the greatest lower bound of θ-values satisfying (4.3a) and (4.3b)$\}$.

Thus, τ is a function of marginal parameters and ω. The range of possible values of θ is given in Lemma 4.2.

Lemma 4.2. *Suppose that* $\theta \leq \omega f_{X_1}(0) + f_{X_2}(0)$. *If*

$$\theta \in [max\{\tau, max(\omega f_{X_1}(0), f_{X_2}(0))\}, \omega f_{X_1}(0) + f_{X_2}(0)] \qquad (4.5)$$

then Theorem 4.1 is verified. Moreover, the joint distribution of (X_1, X_2) *given by (3.3) is absolutely continuous if and only if* $\theta = \omega f_{X_1}(0) + f_{X_2}(0)$.

Proof. We know from the proof of Theorem 4.1 that

$$P(X_1 > \omega X_2) = 1 - \frac{\omega f_{X_1}(0)}{\theta} \geq 0 \quad \text{and hence} \quad \omega f_{X_1}(0) \leq \theta.$$

By analogy, from the expression for $P(X_1 < \omega X_2)$ we obtain $f_{X_2}(0) \leq \theta$. Thus, $\theta \geq \max\{\omega f_{X_1}(0), f_{X_2}(0)\}$.

Depending on the parameters of marginal densities, it may happen that $\tau > \omega f_{X_1}(0) + f_{X_2}(0)$ in some special cases, which would contradict inequalities (4.2). Such high τ-values would be outside of the parameter space of W-BLMP. The lower bound in (4.5) can be obtained by checking the restrictions on θ imposed by inequalities (4.3a) and (4.3b) in order to get the associated value of τ.

When $\alpha = P(X_1 = \omega X_2) = 0$, then $\omega f_{X_1}(0) + f_{X_2}(0) - \theta = 0$. Since $\alpha \in [0, 1]$ and $\omega f_{X_1}(0) + f_{X_2}(0)$ is increasing function on ω we always have $\theta \leq \omega f_{X_1}(0) + f_{X_2}(0)$, which is the upper bound for θ in (4.5).

Finally, observe that the joint density function is absolutely continuous if and only if $\alpha = 0$, i.e. when $\theta = \omega f_{X_1}(0) + f_{X_2}(0)$. This completes the proof. $\qquad \Box$

Denote by $\Theta = (\theta, \omega, \Lambda)$ the parameter vector of W-BLMP class of distributions, where Λ is the vector of parameters of marginal distributions. The parameter space of W-BLMP is well defined when θ belongs to the closed interval given in (4.5). It is crucial for construction of proper bivariate continuous distributions possessing W-BLMP from pre-specified marginal densities as we will see in the next example.

Example 4.1. *Let $c_1, c_2, c_3 > 0$ and the densities $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ be given by*

$$f_{X_1}(x_1) = \left(c_1 - \frac{c_3}{c_2 + x_1^2}\right) \exp\left\{-c_1 x_1 + \frac{c_3}{\sqrt{c_2}} tan^{-1}\left(\frac{x_1}{\sqrt{c_2}}\right)\right\}, \quad x_1 \geq 0,$$

and

$$f_{X_2}(x_2) = \omega\left(c_1 - \frac{c_3}{c_2 + \omega^2 x_2^2}\right) \exp\left\{-c_1 \omega x_2 + \frac{c_3}{\sqrt{c_2}} tan^{-1}\left(\frac{\omega}{\sqrt{c_2}} x_2\right)\right\}, \quad x_2 \geq 0,$$

with $\omega > 0$. In both cases the condition $c_1 \geq \frac{c_3}{c_2}$ ensures that $f_{X_i}(0) \geq 0$, $i = 1, 2$.

On Figure 6 we show the form of density $f_{X_2}(x_2)$ for fixed c_1, c_2, c_3 and four different values (0.78, 1.0, 1.2 and 1.44) of the parameter ω. One can detect the impact of the parameter ω on the skewness and leptokurtosis of the density of X_2.

Consider a joint density $f_{X_1,X_2}(x_1, x_2)$ represented by

$$\begin{cases} \left\{\frac{c_3\omega[2(x_1-\omega x_2)-c_3]}{\left[c_2+(x_1-\omega x_2)^2\right]^2} + \frac{c_3(2c_1\omega-\theta)}{c_2+(x_1-\omega x_2)^2} - c_1\left(c_1\omega - \theta\right)\right\} \\ \times \exp\left\{-\left[c_1 x_1 + (\theta - c_1\omega)x_2 - \frac{c_3}{\sqrt{c_2}} tan^{-1}\left(\frac{x_1-\omega x_2}{\sqrt{c_2}}\right)\right]\right\}, \quad \text{if } x_1 \geq \omega x_2, \\ \\ \left\{\frac{c_3\omega[2(\omega x_2-x_1)-c_3]}{\left[c_2+(\omega x_2-x_1)^2\right]^2} + \frac{c_3(2c_1\omega-\theta)}{c_2+(\omega x_2-x_1)^2} - c_1\left(c_1\omega - \theta\right)\right\} \\ \times \exp\left\{-\left[c_1\omega x_2 + \left(\frac{\theta}{\omega} - c_1\right)x_1 - \frac{c_3}{\sqrt{c_2}} tan^{-1}\left(\frac{\omega x_2-x_1}{\sqrt{c_2}}\right)\right]\right\}, \quad \text{if } x_1 \leq \omega x_2. \end{cases}$$

One can verify that $f_{X_1,X_2}(x_1, x_2)$ has marginals $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ indeed. The parameter space in this case is $\Theta = (\theta, \omega, \Lambda)$ with $\Lambda = (c_1, c_2, c_3)$.

Applying (4.4) we obtain that $(1-\alpha)f^{ac}_{X_1,X_2}(x_1, x_2)$ is given by

$$\begin{cases} \theta\left\{\frac{c_3\omega[2(x_1-\omega x_2)-c_3]}{\left[c_2+(x_1-\omega x_2)^2\right]^2} + \frac{c_3(2c_1\omega-1)}{c_2+(x_1-\omega x_2)^2} - c_1\left(c_1\omega - 1\right)\right\} \\ \times \exp\left\{-\left[c_1 x_1 + (\theta - c_1\omega)x_2 - \frac{c_3}{\sqrt{c_2}} tan^{-1}\left(\frac{x_1-\omega x_2}{\sqrt{c_2}}\right)\right]\right\}, \quad \text{if } x_1 \geq \omega x_2, \\ \\ \theta\left\{\frac{c_3\omega[2(\omega x_2-x_1)-c_3]}{\left[c_2+(\omega x_2-x_1)^2\right]^2} + \frac{c_3(2c_1\omega-1)}{c_2+(\omega x_2-x_1)^2} - c_1\left(c_1\omega - 1\right)\right\} \\ \times \exp\left\{-\left[c_1\omega x_2 + \left(\frac{\theta}{\omega} - c_1\right)x_1 - \frac{c_3}{\sqrt{c_2}} tan^{-1}\left(\frac{\omega x_2-x_1}{\sqrt{c_2}}\right)\right]\right\}, \quad \text{if } x_1 \leq \omega x_2. \end{cases}$$

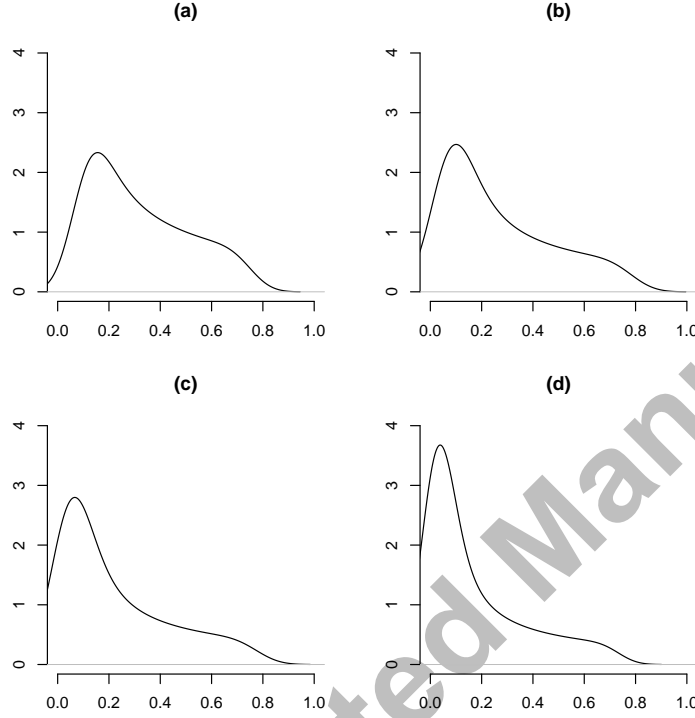*After some algebra we obtain $\alpha = P(X_1 = \omega X_2) = \frac{2\omega}{\theta}(c_1 - \frac{c_3}{c_2}) - 1$.*

Figure 6: Marginal density $f_{X_2}(x_2)$ in Example 4.1 with $c_1 = 1$, $c_2 = 3$ and $c_3 = 1$ for (a) $\omega = 0.78$, (b) $\omega = 1.0$, (c) $\omega = 1.2$ and (d) $\omega = 1.44$.

If $x_1 = \omega x_2 = x \geq 0$, for the singular component of $S_{X_1,X_2}(x_1, x_2)$ from Lemma 4.1 we obtain

$$S_{X_1,X_2}^{si}\left(x, \frac{x}{\omega}\right) = \exp\left(-\frac{\theta x}{\omega}\right) \quad \text{and thus} \quad \alpha f_{X_1,X_2}^{si}\left(x, \frac{x}{\omega}\right) = \frac{\theta x}{\omega} \exp\left(-\frac{\theta x}{\omega}\right).$$

Figure 7 displays the shape of the density $f_{X_1,X_2}(x_1, x_2)$ for the same fixed values of parameters, e.g., $c_1 = 1, c_2 = 3, c_3 = 1$ and ω used in the graphs of Figure 6.

The corresponding lines of singularity L_ω are colored in red. Note that the parameter ω introduces a certain asymmetry on the form of bivariate density especially in the upper tails. When $\omega = 1$, we have $f_{X_1}(x_1) = f_{X_2}(x_2)$ for all $x_1, x_2 \geq 0$. This particular case, see Figure 7 (b), has been considered by Kulkarni (2006)*.*
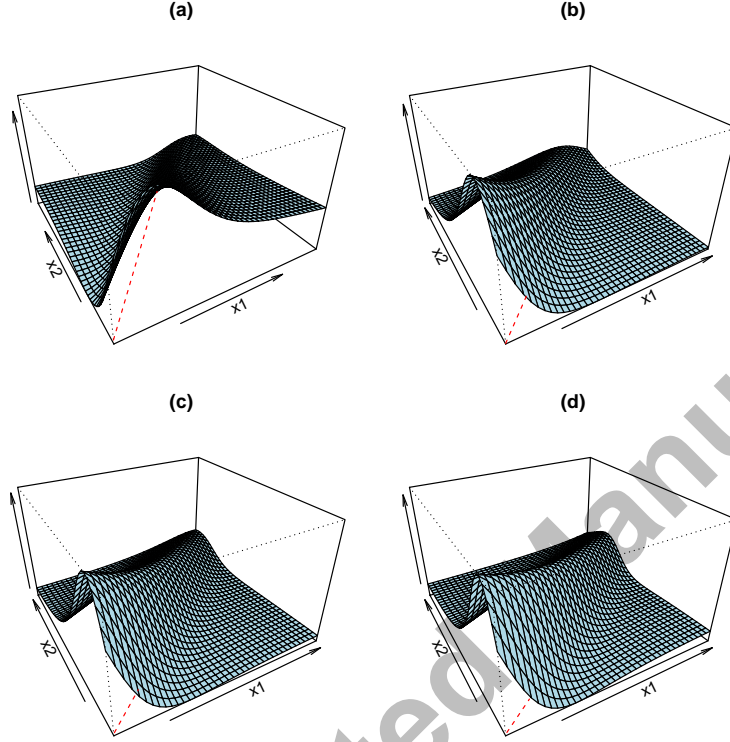
Figure 7: Bivariate density $f_{X_1,X_2}(x_1,x_2)$ with $c_1 = 1$, $c_2 = 3$, $c_3 = 1$ and $\theta = 1$ for (a) $\omega = 0.78$, (b) $\omega = 1.0$, (c) $\omega = 1.2$ and (d) $\omega = 1.44$.

Restrictions (4.3a) and (4.3b) imply that if $x_1 \geq \omega x_2$, then

$$\theta + \frac{2\omega c_3(x_1 - \omega x_2)}{\{c_1[c_2 + (x_1 - \omega x_2)^2] - c_3\}[c_2 + (x_1 - \omega x_2)^2]} - \omega\Big[c_1 - \frac{c_3}{c_2 + (x_1 - \omega x_2)^2}\Big] \geq 0$$

and if $x_1 \leq \omega x_2$ we get

$$\theta + \frac{2\omega c_3(\omega x_2 - x_1)}{\{c_1[c_2 + (\omega x_2 - x_1)^2] - c_3\}[c_2 + (\omega x_2 - x_1)^2]} - \omega\Big[c_1 - \frac{c_3}{c_2 + (\omega x_2 - x_1)^2}\Big] \geq 0.$$

These functions are increasing in x_1 and x_2, with a minimum at the point $(0,0)$. Hence, from definition of τ we have $\tau = \omega(c_1 - \frac{c_3}{c_2})$ and using (4.5) for the possible values of θ we obtain the interval $[\omega(c_1 - \frac{c_3}{c_2}), 2\omega(c_1 - \frac{c_3}{c_2})]$. When $c_1 = 1, c_2 = 3$ and $c_3 = 1$, we obtain $\theta \in [\frac{2\omega}{3}, \frac{4\omega}{3}]$. The graphs on Figure 7 are plotted when $\theta = 1$. We selected this value since it belongs to

the interval $\left[\frac{2\omega}{3}, \frac{4\omega}{3}\right]$ *for all chosen values (0.78, 1.0, 1.2 and 1.44) of the parameter ω, i.e., all bivariate distributions have a singular part.*

If it happens that $\theta = 2\omega(c_1 - \frac{c_3}{c_2})$, the corresponding bivariate distribution will be absolutely continuous according to Lemma 4.1. When $c_1 = 1$, $c_2 = 3$, $c_3 = 1$, we obtain $\theta = \frac{4\omega}{3}$ and Figure 8 shows the shape of absolutely continuous densities corresponding to cases (a) - (d) displayed on Figure 7.
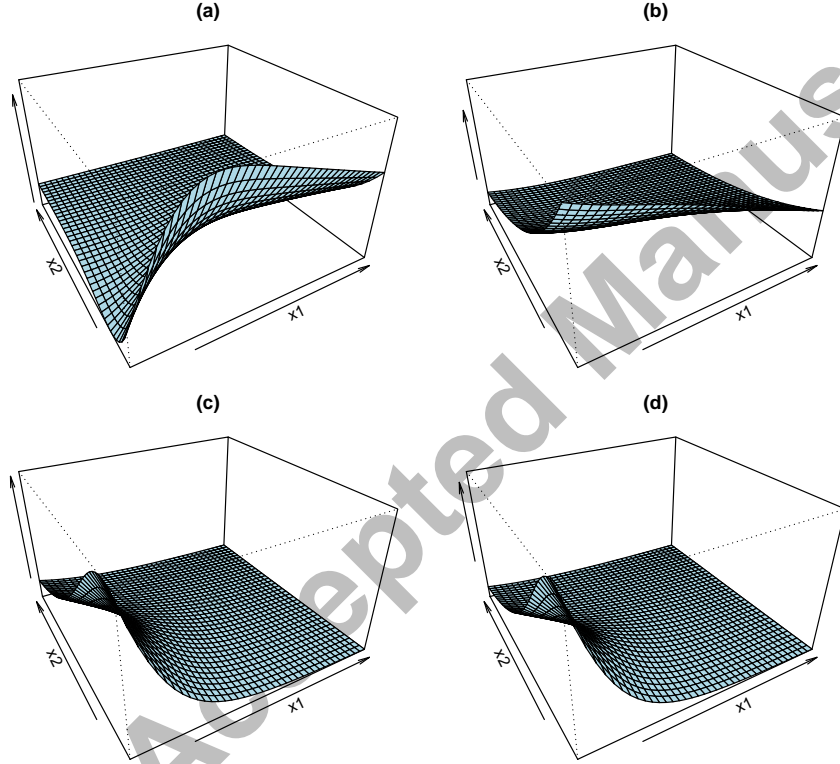

Figure 8: Absolutely continuous $f_{X_1,X_2}(x_1, x_2)$ for $c_1 = 1$, $c_2 = 3$, $c_3 = 1$.
(a) $\omega = 0.78$ and $\theta = 1.04$; (b) $\omega = 1.0$ and $\theta = 1.3334$;
(c) $\omega = 1.2$ and $\theta = 1.6$ and (d) $\omega = 1.44$ with $\theta = 1.92$.

4.2 Marginal failure rate restrictions

Denote by $r_{X_i}(x_i)$ the marginal failure rate of a continuous random variable X_i, i.e. $r_{X_i}(x_i) = \frac{f_{X_i}(x_i)}{S_{X_i}(x_i)}$, $x_i \geq 0$ for $i = 1, 2$. The last equality indicates that the relations from Section 4.1 treating marginal densities can be rewritten in terms of marginal failure rates.

First note that the condition

$$\int_0^\infty r_{X_i}(x)\, dx = \infty, \quad i = 1, 2 \tag{4.6}$$

is necessary for $r_{X_i}(x)$ to be a failure rate, see Barlow and Proschan (1981).

On the other hand, the conditional failure rates $r_i(x_1, x_2)$ introduced in Johnson and Kotz (1975) by $r_i(x_1, x_2) = \frac{\partial}{\partial x_i}\left[-\ln S_{X_1, X_2}(x_1, x_2)\right]$ should be non-negative, $i = 1, 2$. Remind that the hazard vector $(r_1(x_1, x_2), r_2(x_1, x_2))$ uniquely determines the joint distribution of (X_1, X_2) in terms of line integral, see Marshall (1975).

Using (3.3) and conditions $r_i(x_1, x_2) \geq 0$, $i = 1, 2$, we obtain following inequalities

$$0 \leq r_{X_1}(x_1) \leq \frac{\theta}{\omega} \quad \text{and} \quad 0 \leq r_{X_2}(x_2) \leq \theta \quad \text{for all} \quad x_1, x_2 \geq 0. \tag{4.7}$$

Limitations (4.6) and (4.7) indicate that the marginal failure rates $r_{X_1}(x_1)$ and $r_{X_2}(x_2)$ of W-BLMP should

- have unbounded support;
- be restricted from above by the lines $\{\omega x_1 = \theta\}$ and $\{x_2 = \theta\}$, correspondingly.

Additional constraints on marginal failure rates are given in the following statement, being a consequence of Theorem 4.1.

Corollary 4.1. *Let X_i be a non-negative random variable with differentiable failure rates $r_{X_i}(x_i)$ on all finite intervals, $i = 1, 2$. Then $S_{X_1, X_2}(x_1, x_2)$ specified by (3.3) is a proper bivariate survival function if only if*

$$\theta \leq \omega r_{X_1}(0) + r_{X_2}(0) \leq 2\theta \tag{4.8}$$

and

$$\theta - \omega r_{X_1}(x_1 - \omega x_2) + \omega \frac{d}{dx_1}\ln\left[r_{X_1}(x_1 - \omega x_2)\right] \geq 0, \quad for \quad x_1 \geq \omega x_2, \tag{4.9a}$$

$$\theta - r_{X_2}\left(x_2 - \frac{x_1}{\omega}\right) + \frac{d}{dx_2}\ln\left[r_{X_2}\left(x_2 - \frac{x_1}{\omega}\right)\right] \geq 0, \quad for \quad x_1 \leq \omega x_2. \tag{4.9b}$$

Proof. Really, inequalities (4.8) follow from (4.2), since $r_{X_i}(0) = f_{X_i}(0)$ for $i = 1, 2$.

The condition (4.9a) can be obtained from (4.3a), by using the relation

$$\frac{d}{dx_1}[\ln f_{X_1}(x_1 - \omega x_2)] = \frac{d}{dx_1}[\ln r_{X_1}(x_1 - \omega x_2)] - r_{X_1}(x_1 - \omega x_2) \quad \text{for} \quad x_1 \geq \omega x_2.$$

Similarly, (4.9b) is a consequence of (4.3b) when $x_1 \leq \omega x_2$. $\qquad\square$

In terms of failure rates, the interval (4.5) of admissible values of θ transforms into

$$\theta \in [\max\{\tau, \max(\omega r_{X_1}(0), r_{X_2}(0))\}, \omega r_{X_1}(0) + r_{X_2}(0)], \qquad (4.10)$$

where τ is the greatest lower bound of θ-values satisfying (4.9a) and (4.9b).

Notice that if the marginal failure rates are increasing functions, then $\tau = \max(\omega r_{X_1}(0), r_{X_2}(0))$ and therefore relation (4.10) can be written as

$$\theta \in [\max(\omega r_{X_1}(0), r_{X_2}(0)), \omega r_{X_1}(0) + r_{X_2}(0)]. \qquad (4.11)$$

We link above conditions in Theorem 4.2 which enable to generate bivariate distributions possessing W-BLMP from given (pre-specified) marginal failure rates.

Theorem 4.2. *Let the marginal failure rates $r_{X_i}(x_i)$ be differentiable functions. If relations (4.6) to (4.10) hold, then the function $S_{X_1,X_2}(x_1, x_2)$ given by (3.3) is a proper joint survival function having W-BLMP with marginals*

$$S_{X_i}(x) = \exp\left(-\int_0^x r_{X_i}(t)\, dt\right), \quad x \geq 0, \ i = 1, 2. \qquad (4.12)$$

Theorem 4.2 is a counterpart of a similar statement treating BLMP case given by Kulkarni (2006), see her Theorem 3. The converse of Theorem 4.2 also holds for non-degenerate distributions and the statement is given below.

Proposition 4.1. *If $S_{X_1,X_2}(x_1, x_2)$ is non-degenerate bivariate survival function given by equation (3.3) and has differentiable marginal failure rates then it must satisfy conditions (4.6) to (4.10).*

Proof. Follows step by step the proof of Proposition 1 in Kulkarni (2006).
$\qquad\square$

4.2.1 *Examples.* Here we generate bivariate distributions possessing W-BLMP from their given (pre-specified) marginal failure rates $r_{X_1}(x_1)$ and $r_{X_2}(x_2)$ that are increasing or/and decreasing functions. After checking restrictions (4.6) to (4.10), one should use (4.12) to get the marginal survival functions $S_{X_i}(x_i)$, $i = 1, 2$, and then to apply (3.3) in order to obtain the corresponding valid joint survival function $S_{X_1,X_2}(x_1, x_2)$.

Example 4.2. (Example 4.1 continued, with increasing marginal failure rates). *If we adopt the marginal densities $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ from Example 4.1, we will obtain the corresponding marginal failure rates given by $r_{X_1}(x_1) = c_1 - \frac{c_3}{c_2 + x_1^2}$ for $x_1 \geq 0$ and $r_{X_2}(x_2) = \omega\left(c_1 - \frac{c_3}{c_2 + \omega^2 x_2^2}\right)$ for $x_2 \geq 0$ where $c_2, c_3, \omega > 0$ and $c_1 \geq \frac{c_3}{c_2}$. In this case $r_{X_1}(x_1)$ and $r_{X_2}(x_2)$ are increasing functions of their arguments and we will obtain the interval of admissible values of θ from (4.11). Since $r_{X_1}(0) = c_1 - \frac{c_3}{c_2}$ and $r_{X_2}(0) = \omega(c_1 - \frac{c_3}{c_2})$ we have*

$$\theta \in \left[\omega\left(c_1 - \frac{c_3}{c_2}\right), 2\omega\left(c_1 - \frac{c_3}{c_2}\right)\right].$$

Naturally, we got the same interval for possible values of the parameter θ as earlier. Using (3.3) we find the corresponding valid joint survival function

$$S_{X_1,X_2}(x_1,x_2) = \begin{cases} \exp\left\{-\left[c_1 x_1 + (\theta - c_1\omega)x_2 - \frac{c_3}{\sqrt{c_2}}tan^{-1}\left(\frac{x_1 - \omega x_2}{\sqrt{c_2}}\right)\right]\right\}, & \text{if } x_1 \geq \omega x_2, \\ \exp\left\{-\left[c_1\omega x_2 + \left(\frac{\theta}{\omega} - c_1\right)x_1 - \frac{c_3}{\sqrt{c_2}}tan^{-1}\left(\frac{\omega x_2 - x_1}{\sqrt{c_2}}\right)\right]\right\}, & \text{if } x_1 \leq \omega x_2. \end{cases}$$

Example 4.3. (increasing marginal failure rates). *For $x_1, x_2 \geq 0$, $c_1, \omega > 0$ and $0 \leq c_2 \leq 1$, define $r_{X_1}(x_1) = \frac{\exp(c_1 x_1)}{c_2 + \exp(c_1 x_1)}$ and $r_{X_2}(x_2) = \frac{\omega \exp(c_1 x_2)}{c_2 + \exp(\omega c_1 x_2)}$. The marginal failure rates $r_{X_1}(x_1)$ and $r_{X_2}(x_2)$ are increasing functions with $r_{X_1}(0) = \frac{\omega}{1+c_2}$ and $r_{X_2}(0) = \frac{2\omega}{1+c_2}$. Thus, from (4.11) we conclude that $\theta \in \left[\frac{\omega}{1+c_2}, \frac{2\omega}{1+c_2}\right]$. Applying (3.3) we obtain the proper joint survival function*

$$S_{X_1,X_2}(x_1,x_2) = \begin{cases} \exp\left\{-\left[\theta x_2 + \frac{1}{c_1}\ln\frac{c_2 + \exp(c_1 x_1 - c_1 \omega x_2)}{c_2 + 1}\right]\right\}, & \text{if } x_1 \geq \omega x_2, \\ \exp\left\{-\left[\frac{\theta x_1}{\omega} + \frac{1}{c_1}\ln\frac{c_2 + \exp(c_1 \omega x_2 - c_1 x_1)}{c_2 + 1}\right]\right\}, & \text{if } x_1 \leq \omega x_2, \end{cases}$$

and it is absolutely continuous if $\theta = \frac{2\omega}{1+c_2}$, according to Lemma 4.1.

Let us note that the admissible values of the marginal distribution parameters may be further limited as a consequence of inequalities (4.9a) and (4.9b), as illustrated in the next example.

Example 4.4. (increasing and decreasing marginal failure rates). *Let $r_{X_1}(x_1) = \frac{1}{1+x_1}$ and $r_{X_2}(x_2) = \omega(2\omega c_2 x_2 + c_3)$ for $x_1, x_2 \geq 0$ with parameters $c_2, c_3, \omega > 0$. One can verify that $r_{X_1}(x_1)$ is a decreasing and $r_{X_2}(x_2)$ is an increasing function, with $r_{X_1}(0) = 1$ and $r_{X_2}(0) = \omega c_3$. So, the upper bound for θ from (4.10) is $\omega(c_3 + 1)$.*

Inequalities ($4.9a$) and ($4.9b$) imply that $\theta \geq 2\omega$ and $\theta \geq \frac{\omega(c_3^2-2c_2)}{c_3}$. Therefore, $\tau = \frac{\omega(c_3^2-2c_2)}{c_3}$ when $\frac{(c_3^2-2c_2)}{c_3} \leq 2$ and we obtain the following additional restrictions of parameters: $2c_2 \geq c_3(c_3-2)$ and $c_3 \geq 2$. Thus, the admissible values of θ belong to the interval $\left[\max \omega\left(\frac{c_3^2-2c_2}{c_3}, c_3\right), \omega(1+c_3)\right]$.

With the help of (3.3) one can get the expression for the corresponding joint survival function. Finally, if $\theta = \omega(c_3 + 1)$ the joint distribution is absolutely continuous.

On Figure 9 we show the shape of the joint density when it has a singular component and in absolutely continuous case (the singularity line $x_1 = 1.2x_2$ is given in red), see graphs (a) and (b), correspondingly.
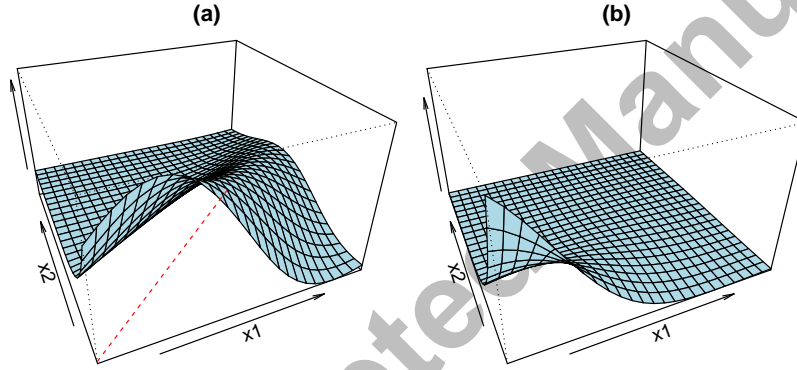

Figure 9: Density $f_{X_1,X_2}(x_1, x_2)$ in Example 4.4 for $c_2 = 4.0$ and $c_3 = 5.0$.
(a) $\omega = 1.2$ and $\theta = 5.0$ (with singular part);
(b) $\omega = 1.2$ and $\theta = 6.0$ (absolutely continuous).

4.2.2 *Distributions with independent marginals.* Let the marginals X_1 and X_2 of continuous joint distribution (X_1, X_2) be independent. In terms of conditional failure rates, the independence between X_1 and X_2 implies that

$$r_i(x_1, x_2) = \frac{\partial}{\partial x_i}\Big[-\ln S_{X_1,X_2}(x_1, x_2)\Big] = r_{X_i}(x_i), \quad i = 1, 2,$$

i.e., $r_1(x_1, x_2) + r_2(x_1, x_2) = r_{X_1}(x_1) + r_{X_2}(x_2)$. Assume further that (X_1, X_2) possesses W-BLMP, e.g., $S_{X_1,X_2}(x_1, x_2)$ is given by (3.3) and we obtain

$$r_{X_1}(x_1) + r_{X_2}(x_2) = \begin{cases} \theta + (1-\omega)r_{X_1}(x_1 - \omega x_2), & \text{if} \quad x_1 \geq \omega x_2, \\ \frac{\theta}{\omega} + (1-\frac{1}{\omega})r_{X_2}(x_2 - \frac{x_1}{\omega}), & \text{if} \quad x_1 \leq \omega x_2. \end{cases}$$

Let $x_1 \geq \omega x_2$. Then equation

$$r_{X_1}(x_1) + r_{X_2}(x_2) = \theta + (1-\omega)r_{X_1}(x_1 - \omega x_2) \tag{4.13}$$

is valid for all $x_1, x_2 \geq 0$ and $\omega > 0$. Substitute $x_2 = 0$ in (4.13) to conclude that $\omega r_{X_1}(x_1) = \theta - r_{X_2}(0)$ for all $x_1 \geq 0$. Thus, the failure rate $r_{X_1}(x_1)$ is a positive constant $\frac{\theta - r_{X_2}(0)}{\omega}$ for all $x_1 \geq 0$ and therefore X_1 is exponentially distributed with a parameter $\frac{\theta - r_{X_2}(0)}{\omega}$.

Now set $x_1 = x_2 = 0$ in (4.13) to get $\theta = \omega r_{X_1}(0) + r_{X_2}(0)$, which is the upper bound in (4.11), meaning that the corresponding joint distribution is absolutely continuous according to Lemma 4.1.

By analogy, if $x_1 \leq \omega x_2$ we obtain that X_2 is exponentially distributed with a parameter $\theta - \omega r_{X_1}(0)$. Similarly, the corresponding joint distribution should be absolutely continuous. Hence, we proved the following

Corollary 4.2. *If a joint distribution (X_1, X_2) with independent marginals possesses W-BLMP, then it is absolutely continuous. Moreover, the marginals X_1 and X_2 are exponentially distributed.*

The proof of Corollary 4.2 indicates that if X_1 has an exponential distribution with a parameter $\lambda_1 > 0$, then X_2 is exponentially distributed with a parameter $\lambda_2 = \theta - \omega \lambda_1 > 0$. Thus, the corresponding joint absolutely continuous distribution with independent marginals is given by

$$S_{X_1,X_2}(x_1, x_2) = \exp\{-\lambda_1 x_1 - (\theta - \omega \lambda_1)x_2\}. \tag{4.14}$$

4.2.3 *Closure properties.* Let $\mathcal{W}(x_1, x_2; \Theta)$ be the collection of proper bivariate distributions possessing W-BLMP and their marginals satisfying Theorem 4.2. Denote by $\Theta = (\theta, \omega, \Lambda)$ the corresponding (generic) parameter vector, with Λ being the vector of all parameters of marginal distributions. Note that Θ will be different under update of parameters θ, ω or Λ. We will present in the next statement several closure properties of the class $\mathcal{W}(x_1, x_2; \Theta)$.

Theorem 4.3. *Denote by $\mathcal{S}, \mathcal{S}_1$ and $\mathcal{S}_2$ survival functions belonging to the class $\mathcal{W}(x_1, x_2; \Theta)$. The following closure properties are fulfilled:*

(CP1) *If $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{W}(x_1, x_2; \Theta)$ then their product $\mathcal{S}_1 \mathcal{S}_2 \in \mathcal{W}(x_1, x_2; \Theta)$;*
(CP2) *If $\mathcal{S} \in \mathcal{W}(x_1, x_2; \Theta)$ then $[\mathcal{S}]^c \in \mathcal{W}(x_1, x_2; \Theta)$ for some $c \geq 1$;*
(CP3) *If $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{W}(x_1, x_2; \Theta)$ then $[\mathcal{S}_1]^{c_1}[\mathcal{S}_2]^{c_2} \in \mathcal{W}(x_1, x_2; \Theta)$ for some $c_1, c_2 \geq 1$;*
(CP4) *If $S_{Y_1,Y_2}(x_1, x_2)$ belongs to $\mathcal{W}(x_1, x_2; \Theta)$ and $\beta > 0$, then $S_{X_1,X_2}(x_1, x_2) = S_{Y_1,Y_2}(\beta x_1, \beta x_2)$ also belongs to $\mathcal{W}(x_1, x_2; \Theta)$;*
(CP5) *If $\mathcal{S}_1, \mathcal{S}_2$ have the same parameter vector Θ and belong to $\mathcal{W}(x_1, x_2; \Theta)$, then $\frac{\mathcal{S}_1 + \mathcal{S}_2}{2} \in \mathcal{W}(x_1, x_2; \Theta)$.*

Proof. The properties (CP1) to (CP5) follow after applying relation (3.3) for the corresponding case. □

Remark 4.2. Applications of closure properties. *The closure properties listed in Theorem 4.3 enable us to construct plenty of bivariate distributions belonging to $\mathcal{W}(x_1, x_2; \Theta)$ by using as a base a known distribution defined by (3.3), being an output of Theorem 4.2. Moreover, relations (CP1), (CP3) and (CP5) can be easily extended to any finite number of survival functions belonging to $\mathcal{W}(x_1, x_2; \Theta)$.*

Note that if the vector (Y_1, Y_2) is independent of the vector (Z_1, Z_2) and their survival functions belong to $\mathcal{W}(x_1, x_2; \Theta)$ class, the implication (CP1) means that the survival function of the vector

$$(X_1, X_2) = [\min(Y_1, Z_1), \min(Y_2, Z_2)]$$

also belongs to $\mathcal{W}(x_1, x_2; \Theta)$. This fact has applications in series systems.

We finish with a complementary closure property given in the following claim.

Lemma 4.3. *If $S_{T_1, T_2}(x_1, x_2)$ belongs to $\mathcal{W}(x_1, x_2; \Theta)$ and T_3 is independent of (T_1, T_2) with a survival function*

$$S_{T_3}(\max(x_1, \omega x_2)) = \begin{cases} \exp\{-\lambda \omega x_2\}, & \text{if } x_1 \geq \omega x_2, \\ \exp\{-\lambda x_1\}, & \text{if } x_1 \leq \omega x_2 \end{cases} \qquad (4.15)$$

for all $x_1, x_2 \geq 0$ and $\lambda > 0$, then stochastic relation (2.1) generates a joint distribution belonging to the class $\mathcal{W}(x_1, x_2; \Theta)$.

Proof. Let $S_{T_1, T_2}(x_1, x_2)$ be specified by (3.3), e.g., $(T_1, T_2) \in \mathcal{W}(x_1, x_2; \Theta)$ with $\Theta = (\theta, \omega, \Lambda)$. Using stochastic representation (2.1) we obtain

$$S_{X_1, X_2}(x_1, x_2) = S_{T_1, T_2}(x_1, x_2) \, S_{T_3}(\max(x_1, \omega x_2)).$$

Applying (4.15) we get

$$S_{X_1, X_2}(x_1, x_2) = \begin{cases} \exp\{-(\theta + \lambda \omega)x_2\}S_{X_1}(x_1 - \omega x_2), & \text{if } x_1 \geq \omega x_2, \\ \exp\{-\frac{(\theta + \lambda \omega)x_1}{\omega}\}S_{X_2}(x_2 - \frac{x_1}{\omega}), & \text{if } x_1 \leq \omega x_2. \end{cases}$$
$$(4.16)$$

Therefore, we remain in the class $\mathcal{W}(x_1, x_2; \Theta)$, but now $\Theta = (\theta_1, \omega, \Lambda)$ with $\theta_1 = \theta + \lambda \omega$. □
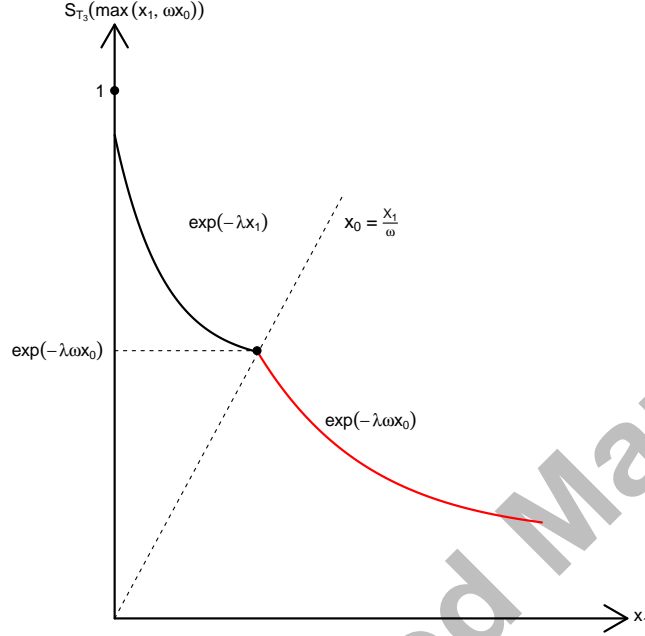
Figure 10: Shape of $S_{T_3}(\min(x_1, \omega x_2))$ for $\omega = 0.5$, $\lambda = 1.0$ and $x_0 = 1.8$.

Observe that distribution (4.15) exhibits a cusp along the line $\{x_1 = \omega x_2\}$, as shown on Figure 10, representing the corresponding "level curves" of S_{T_3} for fixed $x_2 = x_0 = 1.8$, when $\omega = 0.5$ and $\lambda = 1.0$.

Recall that Pinto and Kolev (2015) introduced an Extended MO model with singularity along the line $\{x_1 = x_2\}$ in R_+^2. The important consequence of Lemma 4.3 is that it gives us an idea how to define a modified Extended MO model, but with a possible singularity along the line $L_\omega : \{x_1 = \omega x_2\}$.

5 Discussion and possible further investigations

In this article we introduced a weak version of the classical BLMP property via functional equation (1.5) generated by the stochastic representation (2.1). The additional parameter ω is the inclination of arbitrary line in R_+^2 passing through the origin. The examples illustrating our approach indicate that the asymmetry parameter ω plays a role of a *skew parameter* and implies a delayed effect of the common shock affecting one of the items of a system. As an interpretation, the W-BLMP preserves the distribution

of random vector (X_1, X_2) considering its dynamic in direction of the line $L_\omega : \{x_2 = \omega x_1\}$, instead of keeping it unchanged along the main diagonal, as postulated by BLMP.

The W-BLMP is characterized by the special form of joint survival function $S_{X_1, X_2}(x_1, x_2)$ given by (3.3) which is a solution of functional equation (1.5). This fact is a contribution to the theory of functional equations. The proof of Theorem 3.1 is different from existing approaches treating the BLMP case. We use the fact that the independent solution (3.2) of functional equation (3.1) may serve as a particular solution of (1.5). Finally, we suggest an algorithm for building distributions possessing W-BLMP starting from pre-specified marginal distributions that must satisfy a list of restrictions and enrich it by closure properties of the new BLMP notion.

We will list below several related research problems.

- One would be successful to use our methodology when the support of the singular component lies along a monotonic curve different than the line through the origin. The choice of the model should be influenced on the data at hand;
- Of course, depending on the situation, one might consider a dual version of the W-BLMP replacing the *min* by *max* operation in stochastic representation (2.1). Now, the interest will be related to model our data by the joint distribution function of (X_1, X_2) instead of joint survival one (when we have right censoring, for instance);
- Practical needs indicate that it makes sense to define models with restricted domain, but not "for all $x_1, x_2 \geq 0$" as one usually assumes as a standard. For example, one can still use as a base the functional equation (1.5), but when its arguments belong to a fixed figure, indicated by corresponding scatter plot, a square say. In such a case we will deal with a weaker version of the W-BLMP;
- The "delayed" effect might be identified in the form of a fixed time interval, say "d", instead of the multiplicative factor ω assumed in this paper. Of course, one could start with a fixed value of "d" and, in a second step, to plug-in "d" as a parameter of the joint distribution, as we did with the parameter ω. Such kind of delayed effect would be graphically represented as a shift (upwards or downwards) of the usual support of the singular component, keeping the 45 degree inclination;
- We see promising application of Sibuya's dependence function, being the ratio of joint distribution and the product of marginal distributions, see more details in Pinto and Kolev (2016). The idea is to investigate under which conditions the Sibuya's dependence function is invariant along the line L_ω, i.e., when the Sibuya's dependence func-

tion of (X_1, X_2) and its ω-type residual lifetime vector $\mathbf{X}_{w,t}$ are the same for all $t > 0$.

- In Lemma 4.3 we got a modification of extended Marshall-Olkin distribution given by (4.16) which is based on stochastic representation (2.1). The dependence structure between two items of a system is explained not only by the common shock identified by the random variable T_3, but also by the joint distribution of the individual shocks represented by T_1 and T_2. With this additional source of dependence extended Marshall-Olkin's distributions allow modeling both positive and negative quadrant dependence between its components. Moreover, the model may be non-exchangeable even if the marginals have the same distribution, consult Pinto and Kolev (2015) for details.

The classical MO model (1.2) is probably one of the mostly used in Reliability, Survival analysis and Life Insurance as states Singpurwalla (2006). Hence, any its extension has theoretical and practical importance. We expect that the modified MO models introduced in this article may serve as an alternative to existing ones for analysis of specific data sets exhibiting singularity.

Acknowledgements

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