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The equivalence of dynamic and static asset allocations under the uncertainty caused by Poisson processes

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Abstract. We investigate the equivalence of dynamic and static asset allocations in the case where the price process of a risky asset is driven by a Poisson process. Under some mild conditions, we obtain a necessary and sufficient condition for the equivalence of dynamic and static asset allocations. In addition, we provide a simple sufficient condition for the equivalence.

1 Introduction

Consider a frictionless market which consists of a default-free bond and a risky asset. Assume that the price process B of the bond follows $(\exp(-r(T-t)), t \geq 0)$, where r is the interest rate and T is the maturity date, and the price process S of the risky asset obeys a semimartingale S defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$.

We invest in the bond and the risky asset by *admissible* trading strategies (cf. [5, p. 161, Definition 7.25]) and optimize our expected utility of the terminal wealth. Which strategy is optimal? To answer this question, we solve the optimal problem

$$\begin{cases} \sup \mathbb{E}[U(\phi^{(1)}(T)B(T) + \phi^{(2)}(T)S(T))], \\ \text{s.t. } \phi^{(1)}(0)B(0) + \phi^{(2)}(0)S(0) = W_0 \text{ and } \phi \in \mathcal{A}. \end{cases} \tag{1.1}$$

Here, U is the utility function, W_0 is the initial wealth, and $\mathcal{A}$ is the collection of all admissible trading strategies.

Definition 1.1. The optimal problem (1.1) is called a *dynamic* asset allocation problem.

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Merton [8] exploited a dynamic asset allocation problem in the case where the price process of the risky asset follows a geometry Brownian motion by applying the method of dynamic programming. In [2], under the assumption that the price process of the risky asset follows an Itô diffusion process, Cox and Huang used the martingale method to solve a dynamic asset allocation problem. In [7], Liu and Pan discussed a dynamic asset allocation problem in the case where the price process of the risky asset satisfies a jump diffusion. Pliska [10] obtained a necessary and sufficient condition for the terminal wealth to be optimal.

Assume that the market also includes European options written on the risky asset. Furthermore, we can invest in not only the bond and the risky asset but also the European options, and we are buy-and-hold investors, that is, we cannot trade the risky asset, the bond and options after we buy them at the initial time. What portfolio maximizes our expected utility of the terminal wealth? We answer this question by solving the optimal problem

$$\begin{cases} \sup \mathbb{E}\left[U(W)\right], \\ \text{s.t. } W \in \mathcal{D}_T(W_0). \end{cases} \tag{1.2}$$

Here,

$$\mathcal{D}_T(W_0) := \{\text{The maturity payoffs of all European options written on the risky asset with maturity date } T \text{ and initial price } W_0\}.$$

We have the following definition.

Definition 1.2. The optimal problem (1.2) is called a *static* asset allocation problem.

By the definitions of dynamic and static asset allocation problems, in a complete market, the optimum of a static asset allocation problem cannot be better than that of the corresponding dynamic asset allocation problem. However, as Haugh and Lo [4] and Kohn and Papazoglu [6] pointed out, it may be possible that the dynamic and static asset allocations are equivalent under the assumption that the price process of the risky asset is driven by a Brownian motion, that is, the optimal terminal wealth of a dynamic asset allocation problem can be given by the maturity payoff of a European option.

The following reasons motivate us to study the equivalence of dynamic and static asset allocations. First, the continuous asset allocation cannot be realized in practice. Second, the static asset allocation permits us to invest in European options written on the risky asset, and only requires trading at the initial time. Third, it was shown in [9] that any European option can be approximated by a portfolio consisting of European call options, thus static asset allocations can be realized (nearly) in practice. Last but not least, the similar problem also arises in the hedging of options. Tompkins [13] compared dynamic and static hedges via simulation, while Engelmann *et al.* [3] provided an empirical comparison between them.

Since the terminal payoffs in (1.1) are replicable, we are interested in the equivalence when the market is complete. By the martingale representation theorem, the complete Lévy market is the one where the price process of the risky asset is driven by a Brownian motion or a Poisson process [11, p. 77] and [1, p. 333]. Tankov and Voltchkova [12] showed the same result by requiring the residual hedging error to be zero. Haugh and Lo [4] and Kohn and Papazoglu [6] have investigated the equivalence for the Brownian case. In this paper, we study the equivalence for the Poisson case.

The rest of the paper is organized as follows. In Section 2, under the assumption that the price process of the risky asset is driven by a Poisson process, we characterize the optimal terminal wealth of a dynamic asset allocation problem. In Section 3, we obtain a necessary and sufficient condition for the equivalence of dynamic and static asset allocations (Theorem 3.1). Then we provide a simple sufficient condition for the equivalence (Theorem 3.3). Some conclusions will be drawn in Section 4.

2 The Characteristic of the Optimal Terminal Wealth

This section serves as characterizing the optimal terminal wealth when the price process of the risky asset is driven by a Poisson process.

Let N be a Poisson process with intensity λ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $\widetilde{N}$ be its compensated process. Let $\{\mathscr{F}_t^N\}_{t\geq 0}$ is the augmentation of the natural filtration generated by the process N.

Assume that the price process S of the risky asset satisfies the equation

$$\mathrm{d}S(t) = S(t-)\left(\alpha(t-, S(t-))\mathrm{d}t + \gamma(t-, S(t-))\mathrm{d}\widetilde{N}(t)\right), \tag{2.1}$$

where $\alpha : \overline{\mathbb{R}^+} \times \mathbb{R} \to \mathbb{R}$ and $\gamma : \overline{\mathbb{R}^+} \times \mathbb{R} \to \mathbb{R}$ are two given functions.

We first look for a numeraire pair (cf. [5, p. 156, Definition 7.18]). To this end, we make the following conditions.

- (S1) The functions α and γ are bounded.
- (S2) The functions $x \mapsto \alpha(t, x)x$ and $x \mapsto \gamma(t, x)x$ are Lipschitz continuous uniformly with respect to t.
- (S3) $\frac{r-\alpha(t,x)}{\gamma(t,x)} > -\lambda$, $\gamma(t, x) > -1$ and $\gamma(t, x) \neq 0$ for any $(t, x) \in \overline{\mathbb{R}^+} \times \mathbb{R}$.

Let $Y := (Y(t), 0 \le t \le T)$ be a Lévy type stochastic integral

$$\mathrm{d}Y(t) = a(t-)\mathrm{d}t + c(t-)\mathrm{d}\widetilde{N}(t), \quad Y(0) = 0, \tag{2.2}$$

where

$$a(t) := \lambda \log\left(1 + \frac{r - \alpha(t, S(t))}{\lambda\gamma(t, S(t))}\right) - \frac{r - \alpha(t, S(t))}{\gamma(t, S(t))}$$

and

$$c(t) := \log\left(1 + \frac{r - \alpha(t, S(t))}{\lambda\gamma(t, S(t))}\right).$$

Then from [1, p. 373, Exercise 6.2.5] it follows that $(\exp(Y(t)), 0 \le t \le T)$ is a square-integrable martingale on the filtrated probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t^N\}_{t\ge 0}, \mathbb{P})$.

Let $\mathbb{Q}$ is the probability measure on $(\Omega, \mathscr{F}_T)$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathscr{F}_t} = \exp(Y(t))$, $0 \le t \le T$, then, after some straightforward calculations, we find that $(B, \mathbb{Q})$ is a numeraire pair.

Lemma 2.1. *Assume that*

(U1) *The utility function U is continuously differentiable and strictly concave.*
(U2) $\lim_{w\to-\infty} U'(w) = +\infty$ *and* $\lim_{w\to+\infty} U'(w) = 0$.
(U3) *There are positive numbers β and ε such that $\varepsilon < U'(0) < \beta$.*

Then $(U')^{-1}(kZ)$ is the optimal terminal wealth of the dynamic asset allocation problem (1.1), where k is some positive number, and $Z := \exp(Y(T))$.

Proof. The Lemma follows directly from the results in [10, Theorem 13]. □

3 Conditions for the Equivalence

In this section, we first provide a necessary and sufficient condition for the equivalence of dynamic and static asset allocations. Then we show a simple sufficient condition for the equivalence.

Theorem 3.1. *Problems (1.1) and (1.2) are equivalent if and only if there exists a function g such that $Y(t) = g(t, S(t))$ for $0 \le t \le T$.*

Proof. 1. Note that $\gamma(t, x) \ne 0$ for all $(t, x) \in [0, +\infty) \times \mathbb{R}$. Then from Equation (2.1), we have $\mathscr{F}_t^N = \mathscr{F}_t^S$, where $\{\mathscr{F}_t^S\}_{t\ge 0}$ is the augmentation of the natural filtration generated by the process S.

2. Since the process $(\exp(Y(t)), 0 \le t \le T)$ is a martingale on the filtrated probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t^N\}_{t\ge 0}, \mathbb{P})$, we see that

$$\exp(Y(t)) = \mathbb{E}[\exp(Y(T))|\mathscr{F}_t^N].$$

Noting that problems (1.1) and (1.2) are equivalent if and only if the optimal terminal wealth of (1.1) is a function of $S(T)$, we have $\exp(Y(T)) = f(S(T))$ for some function f by Lemma 2.1. Then it follows that

$$\begin{aligned}\exp(Y(t)) &= \mathbb{E}[f(S(T))|\mathscr{F}_t^N]\\ &= \mathbb{E}[f(S(T))|\mathscr{F}_t^S]\\ &= \mathbb{E}[f(S(T))|S_t]\\ &= h(t, S(t)) \quad \text{for some function } h,\end{aligned}$$

where we have used the Markov property of the process S for the third equality and Doob-Dynkin lemma for the last equality.

Thus we get $Y(t) = \log(h(t, S(t)))$.

3. Suppose that there exists a function g such that $Y(t) = g(t, S(t))$ for $0 \le t \le T$. Then by Lemma 2.1, we find that the optimal terminal wealth of the problem (1.1) is a function of $S(T)$. Hence, problems (1.1) and (1.2) are equivalent. □

Example 3.2. Let $\alpha := r - \lambda$ and $\gamma := 1$. After some direct calculations, we get

$$\begin{aligned}S(t) &= S(0)\exp\left[(r - 2\lambda + \lambda\log 2)t + (\log 2)\widetilde{N}(t)\right] \text{ and}\\ Y(t) &= (\lambda\log 2 - \lambda)t + (\log 2)\widetilde{N}(t).\end{aligned}$$

Therefore, it follows that $Y(t) = (\lambda - r)t + \log(S(t)) - \log(S(0))$, and then problems (1.1) and (1.2) are equivalent. Also refer to Corollary 3.5.

Theorem 3.3. *Problems (1.1) and (1.2) are equivalent if there is a function $g(t,p)$ with $g(0, S(0)) = 0$ such that*

$$\begin{aligned}&\frac{\partial g}{\partial t}(t,p) + \alpha(t,p)p\frac{\partial g}{\partial p}(t,p) - \lambda\gamma(t,p)p\frac{\partial g}{\partial p}(t,p)\\ &= \frac{\alpha(t,p) - r}{\gamma(t,p)},\end{aligned} \tag{3.1}$$

and

$$\log\left(1 + \frac{r - \alpha(t,p)}{\lambda\gamma(t,p)}\right) = g(t, p + \gamma(t,p)p) - g(t,p). \tag{3.2}$$

Proof. 1. Recall the process Y defined by (2.2):

$$\mathrm{d}Y(t) = a(t-)\mathrm{d}t + c(t-)\mathrm{d}\widetilde{N}(t), \quad Y(0) = 0,$$

where

$$a(t) = \lambda \log\left(1 + \frac{r - \alpha(t, S(t))}{\lambda\gamma(t, S(t))}\right) - \frac{r - \alpha(t, S(t))}{\gamma(t, S(t))},$$

and

$$c(t) = \log\left(1 + \frac{r - \alpha(t, S(t))}{\lambda\gamma(t, S(t))}\right).$$

2. Suppose that there is a function g satisfying (3.1) and (3.2).
Recall Equation (2.1):

$$\mathrm{d}S(t) = S(t-)\Big(\alpha(t-, S(t-))\mathrm{d}t + \gamma(t-, S(t-))\mathrm{d}\widetilde{N}(t)\Big).$$

Then, by Itô's formula (see [1, pp.251–252, Theorem 4.4.7]), we have

$$\mathrm{d}g(t, S(t)) = \hat{a}(t-)\mathrm{d}t + \hat{c}(t-)\mathrm{d}\widetilde{N}(t),$$

where

$$\begin{aligned}\hat{a}(t) = {} & \frac{\partial g}{\partial t}(t, S(t)) + \alpha(t, S(t))S(t)\frac{\partial g}{\partial p}(t, S(t)) \\ & + \lambda\Big(g(t, S(t) + \gamma(t, S(t))S(t)) - g(t, S(t)) \\ & \qquad - \gamma(t, S(t))S(t)\frac{\partial g}{\partial p}(t, S(t))\Big),\end{aligned}$$

and

$$\hat{c}(t) = g(t, S(t) + \gamma(t, S(t))S(t)) - g(t, S(t)).$$

Since g satisfies (3.1) and (3.2), we find

$$a(t) = \hat{a}(t) \text{ and } c(t) = \hat{c}(t).$$

Thus the processes $(Y(t), 0 \le t \le T)$ and $(g(t, S(t)), 0 \le t \le T)$ satisfy the same SDE. Consequently, by the uniqueness of the solution, we get $Y(t) = g(t, S(t))$ for $0 \le t \le T$. Thus problems (1.1) and (1.2) are equivalent by Theorem 3.1. □

Corollary 3.4. *If $\alpha = r$, then problems (1.1) and (1.2) are equivalent.*

Proof. If $\alpha = r$, then $g \equiv 0$ satisfies (3.1) and (3.2), and the conclusion follows. □

Corollary 3.5. *Assume that α and γ do not depend on the value of the risky asset. If $\alpha + \lambda\gamma^2 = r$, then problems (1.1) and (1.2) are equivalent.*

Proof. Define the function g by

$$g(t, p) := -\int_0^t \alpha(s)\mathrm{d}s + \log(p) - \log(S(0)).$$

After the direct verification, we find that the function g defined as above satisfies the conditions in Theorem 3.3. Thus problems (1.1) and (1.2) are equivalent. □

4 Conclusions

In general, the optimum of a static asset allocation problem cannot be better than that of the corresponding dynamic asset allocation problem. However, it may be possible that the dynamic and static asset allocations are equivalent when the price process of the risky asset is driven by a Brownian motion (cf. [4] and [6]). In this paper, we consider the equivalence of dynamic and static asset allocations for the case that the price process of the risky asset is driven by a Poisson process. Via restricting utility functions and trading strategies, we obtain a necessary and sufficient condition for the equivalence (Theorem 3.1), and also provide a simple sufficient condition for the equivalence (Theorem 3.3). Since the complete Lévy market consists of the one with a pure diffusion and the one with a pure Poisson process (cf. [11, p. 77], [1, p. 333] or [12]), the present paper together with the paper [6] characterize the equivalence of dynamic and static asset allocations for complete Lévy markets. Some similar characterizations of the equivalence are shared by complete Lévy markets; for example, if the growth rate of the risky asset price equals the risk-free rate, the equivalence holds. Besides, differences between the two cases should be pointed out. For instance, the differential equation (3.1), an ingredient of equivalence criteria, is of first order, whereas the corresponding equation is of second order in pure diffusion cases.

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